Realization of Gauge Symmetry

How do we formally distinguish the phases of a gauge theory — i.e., what is the difference between "spontaneously broken" gauge symmetry and "manifest" gauge symmetry?

E.g. in the abelian Higgs model with

$$V(|\phi|) = \frac{1}{2}\left(|\phi|^2 - v^2/2\right)^2$$

and "weak coupling," we like to say that

$$\langle \phi \rangle = \frac{v}{\sqrt{2}} e^{i\theta}$$

— the Higgs field has a "vacuum expectation value" that "breaks" the gauge symmetry.

Strictly speaking, though, this is incorrect. We cannot use the expectation value of a gauge noninvariant local order parameter as a criterion for identifying a phase transition in a gauge theory. (This is called "Elitzur's theorem.")

Using a local order parameter to distinguish ordered and disordered phase is sensible in the case of a global symmetry, but we should recall exactly what it means.

Consider a field theory with action $S[\phi]$, which is invariant under the action on ϕ of a symmetry group G that acts globally

$$\Omega \in G: \quad \phi(x) \to \Omega\, \phi(x)$$

(Ω independent of x). We can imagine introducing a source term J coupled to ϕ that breaks the symmetry (cf. turning on an external magnetic field in an isotropic ferromagnet):

$$S[\phi] \to S[\phi] - J \int \phi$$

(where J is also constant). In the presence of the source, the expectation value of $\phi(x)$ is

$$\langle \phi(x) \rangle_J = \frac{\int d\phi\, e^{(-S + J\int \phi)}\, \phi(x)}{\int d\phi\, e^{(-S + J\int\phi)}}$$

and we may consider the limiting form of the expectation value of $\phi(x)$ as we turn off the source, in a system of infinite volume:

$$\lim_{J\to 0} \lim_{\text{volume}\to\infty} \langle \phi(x) \rangle_J \equiv v.$$

Then if $\Omega v \neq v$, $\Omega \in G$, we say that G is "spontaneously broken."

In fact, if a global symmetry is spontaneously broken, there are degenerate vacua. E.g., if the symmetry is Z_2 under which

$$\phi(x) \to -\phi(x)$$

we have states $|+\rangle, |-\rangle$ where

$$\langle +|\phi(x)|+\rangle = v,$$
$$\langle -|\phi(x)|-\rangle = -v,$$

related by the Z_2 symmetry. One vacuum or the other is selected, depending on whether J approaches zero from positive or negative values.

Of course we can imagine the ground states

$$\frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$$

for which

$$\langle\phi\rangle = 0$$

But it is not correct to say that we can select these states and have manifest Z_2 symmetry. It's not correct because there is a superselection rule

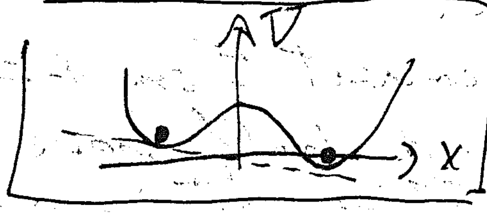
$$0 = \langle +|\mathcal{O}|-\rangle$$

for any local operator $\mathcal{O}$ (the operator cannot flip $\phi(x)$ in an infinite volume). So $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ does not describe a vacuum with $\langle\phi\rangle = 0$; rather it describes an <u>incoherent</u> superposition of the $\langle\phi\rangle = v$ vacuum (w/ probability $\frac{1}{2}$) and the $\langle\phi\rangle = -v$ vacuum (w/ probability $\frac{1}{2}$).

So in our criterion for SSB, we really mean there is <u>some</u> way to take $J \to 0$ limit so that $\langle\phi\rangle \neq 0$ in the limit.

It is also essential to take the Volume $\to \infty$ limit <u>before</u> taking the $J \to 0$ limit. Indeed in 0+1 dimensional spacetime — particle mechanics with $\hbar \neq 0$ or equivalently classical statistical mechanics in 1 spatial dimension with temperature $T \neq 0$ — a discrete symmetry cannot be spontaneously broken.

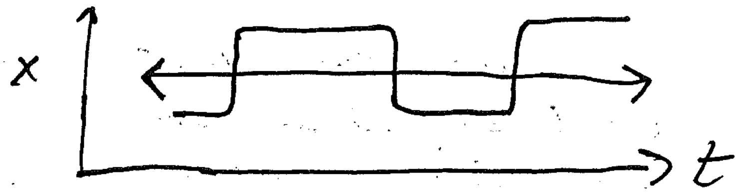
<u>E.g. for a particle</u> in a double well potential:



As $J \to 0$ for $\hbar \neq 0$, there is quantum tunneling through the barrier.

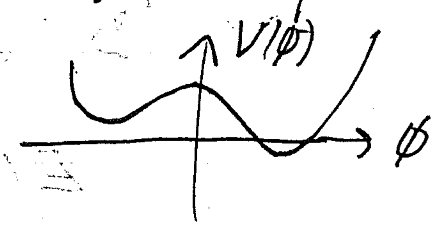
Hence for $J = 0$ there is a <u>unique</u> ground state with $\langle x \rangle = 0$.

From the Euclidean path integral viewpoint, there is a finite density of instantons for $\hbar \neq 0$ as $J \to 0$,

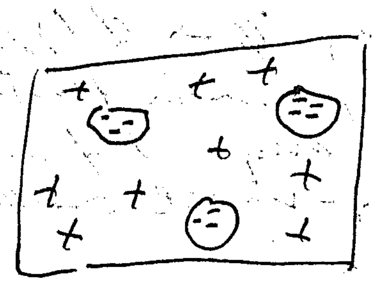


and these always "disorder" the ground state as $t \to \infty$; i.e. + and − values of x become equally likely. This picture also applies to an Ising spin system in one spatial dimension: for $T \neq 0$ there is a nonzero density of "kinks" that disorder the spins.

But in (1+1)-dimensional quantum field theory, or 2-dimensional classical statistical mechanics, SSB of a discrete symmetry is possible. Now for $J \neq 0$, the $\phi = \pm v$ states have different energy densities, and so are split in energy by an ∞ amount in the limit $L \to \infty$. Tunneling to the higher energy state is completely suppressed.



Or -- in the 2-dimensional Ising model in the $\langle \phi \rangle = +v$ phase, small bubbles with $\phi = -v$ can arise as thermal fluctuations. But these carry energy proportional to their surface perimeter, and large bubbles are strongly suppressed by Boltzmann factor $\exp[-\Delta E / T]$.



We say that $D=1$ is the "lower critical dimension" for a discrete global symmetry, meaning that fluctuations can restore the symmetry in $D=1$, but not $D>1$, at any $\hbar \neq 0$ or $T \neq 0$.

For continuous symmetries, things are a little bit different. We still need to take the volume $\to \infty$ limit to have a phase transition, but now the lower critical dimension is $D=2$ ("Mermin-Wagner-Coleman thm"). The fields can fluctuate gradually, so that the gradient energy density remains small.

E.g. a fluctuation of linear size L in which a scalar field ϕ turns by an angle of order one costs energy (in D dimensions)

$$E(L) = \int d^D x (\nabla \phi)^2 \sim v^2 L^{D-2}$$

For $D>2$, these fluctuations are suppressed. But for $D=2$ the energy is scale invariant (unchanged by $\phi \to \phi(\lambda x)$) there is a finite density of such fluctuations on all distance scales for any $T \neq 0$, and these always destroy long-range order.

Now, what about spontaneous breaking of a gauge symmetry? We often introduce gauge fixing in path integral quantization to remove the infinite factor from integrating over the gauge group, but this is not really necessary to define the integral if we have a gauge invariant short-distance regulator (e.g. lattice gauge theory) – the gauge group volume factors out of

$$\langle \theta \rangle : \quad \frac{\int e^{-S}\theta}{\int e^{-S}} \equiv Z^{-1}\int e^{-S}\theta$$

So imagine we have such a regulator, so we can regard the action as gauge invariant:

$$S[\Omega\phi] = S[\phi],$$

where $\Omega(x)$ is a gauge transformation

$$\phi(x) \to \Omega(x)\,\phi(x)$$

(a schematic notation — ϕ represents all fields in the theory $\Omega\phi \equiv \phi^{\Omega}$ indicates their gauge trans properties).

As in our discussion of a global symmetry, consider introducing a source that breaks the local symmetry

$$S[\phi] \to S[\phi] - J\int\phi,$$

and ask if $\langle \phi(x)\rangle$ is gauge-noninvariant in the $J \to 0$ limit:

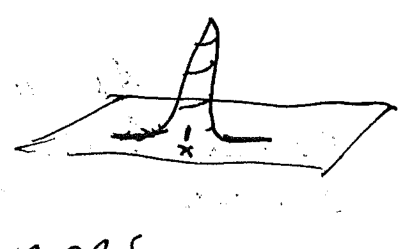
$$\langle \phi(x) \rangle_J = \frac{1}{Z}\int d\phi\, e^{(-S[\phi] + J\int \phi)}\, \phi(x)$$

$$\langle \Omega(x)\, \phi(x) \rangle_J$$
$$= \frac{1}{Z}\int d\phi\, e^{(-S[\phi] + J\int \phi)}\, \Omega(x)\, \phi(x)$$
$$= \frac{1}{Z}\int d\phi\, e^{(-S[\phi] + J\int \Omega^{-1}\phi)}\, \phi(x)$$

– we change variables, using gauge invariance of $S[\phi]$ and $d\phi$. We take the difference, expanding to linear order in J.

$$\langle \phi(x) \rangle - \Omega \langle \phi(x) \rangle = J \left\langle \phi(x) \left[\int dy \left(\phi(y) - \Omega^{-1}(y)\phi(y) \right) \right\rangle_{J=0} + O(J^2)$$

all this would apply also for a global transformation Ω = constant. But now comes the crucial point:
we are free to choose $\Omega(y)$ so that $\Omega(y) = I$ outside of a compact region that contains x
then the y integral converges



$$\int dy \left(\langle \phi(x)\phi(y) \rangle - \langle \phi(x)\, \Omega^{-1}(y)\, \phi(y) \rangle \right) < \infty,$$

and in the $J \to 0$ limit we have

$$\lim_{J\to 0} \langle \phi(x) \rangle_J = \Omega \cdot \lim_{J\to 0} \langle \phi(x) \rangle$$

– the vev of ϕ does not break the symmetry. The (obvious) point really is that if the symmetry is local, taking the volume $\to \infty$ limit has no effect on $\langle \phi(x) \rangle$ because gauge transformations allow $\phi(x)$ to fluctuate locally without any cost in action.

This conclusion does not mean that there can be no phase transitions in a gauge theory, but it does mean that we cannot use the expectation value of a local operator as an order parameter for the transition (e.g. $\langle \phi(x) \rangle$, where $\phi(x)$ is the Higgs field.) This is as expected from the viewpoint of temporal gauge quantization – $\mathcal{H}_{phys}$ is the space of gauge-invariant states, and the observables are the operators that preserve $\mathcal{H}_{phys}$

$$\mathcal{O}: \mathcal{H}_{phys} \to \mathcal{H}_{phys}$$

i.e. gauge-invariant operators. The vacuum, like all physical states is gauge invariant, so there is no possibility of degenerate vacua that are related by "spontaneously broken" gauge transformations.

In e.g. the abelian Higgs model, is there a phase boundary that separates the "Coulomb phase" from the "Higgs phase"? Sure. In the Coulomb phase, the photon mass is $\mu = 0$, in the Higgs phase, $\mu \neq 0$.

There is a phase boundary as we vary the parameter v^2 in the Higgs potential,

Coulomb • Higgs $\rightarrow v$

$\mu_\gamma = 0$ $\quad \mu_\gamma > 0$

where

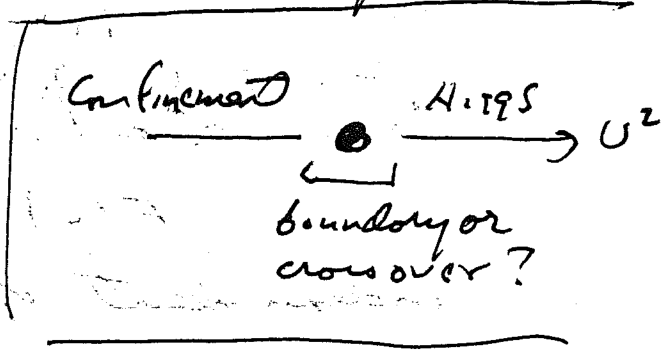
$$V(|\phi|) = \frac{1}{2}\left(|\phi|^2 - v^2/2\right)^2$$

where the photon mass behaves nonanalytically. Another signal (in 2+1 dimensions) is that there is a stable vortex in the particle spectrum of the theory in the Higgs phase, but at the critical point the vortex "disappears" from the spectrum.

But the issue is more subtle if we want to identify a phase boundary separating a Higgs phase from a confinement phase in a gauge theory with gauge group

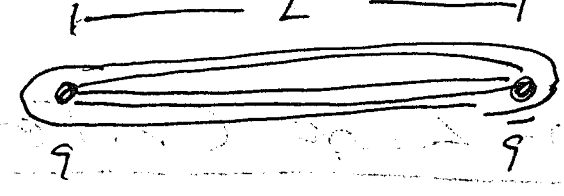
$$G = SU(N)/Z_N$$

e.g. pure Yang-Mills, or Yang-Mills with Higgs fields in the adjoint representation. Here by a "Higgs phase" I mean a phase in which $SU(N)/Z_N$ is "completely Higgsed" — all gluons acquire masses via the Higgs mechanism, so the theory has a mass gap. The "confining phase" also has a mass gap, generated by nonperturbative effects. Are these two phases really different. That is, must there be a sharp boundary, or could there be a gradual crossover from Higgs-like behavior to confinement behavior



We'll argue that there is a boundary, and the appropriate way to probe the phase structure is to consider the response of the vacuum to an external source.

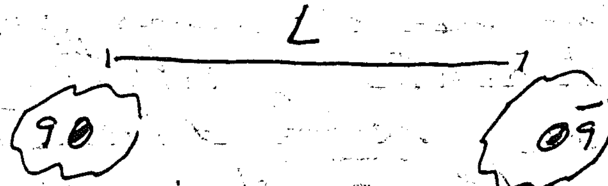
In the $G = SU(N)/\mathbb{Z}_N$ theory, no matter fields transform under $\mathbb{Z}_N$, but we can imagine introducing external classical sources in the fundamental irrep of $SU(N)$. In a confining phase, fundamental



quarks are confined by electric flux tubes, so the vacuum energy in the presence of $q, \bar{q}$ sources separated by distance L will behave as

$$E(L) \approx KL \quad \text{for large } L,$$

where K is the tension (energy per unit length) in the tube.

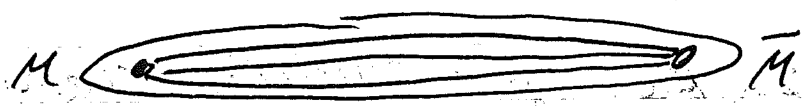


In the Higgs phase, we expect the color of a quark to be <u>screened</u> rather than confined, so that

$$E(L) \to \text{constant}$$

independent of L.

Similarly — there are Z_N strings in the Higgs phase — i.e. monopole sources with Z_N magnetic charge



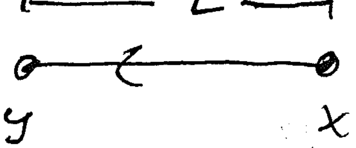
are confined, so the energy $\propto L$. While in the confining phase we anticipate that magnetic charge is screened instead, and Energy $\to$ constant

Wilson Loop Criterion

It is more convenient to express this criterion as an intrinsic property of the gauge-invariant operators of the theory, rather than in terms of the response to external sources.

In fact, our statement about the response to a static color source can be equivalently formulated in terms of the vacuum expectation value of a Wilson loop operator.

We can think of the static color sources as infinitely heavy quarks that are unable to propagate. Introduce an operator that creates a widely separated quark-antiquark pair in a color-singlet state



$$M(y,x) = \bar{q}(y)\, P\exp\left(ig\int_x^y A_\mu dx^\mu\right) q(x)$$

where q annihilates a quark. We need the path ordered exponential to ensure that the operator is gauge invariant, and so creates a color singlet pair. [Physically — we contract the quarks as $\bar{q}^i q_i$ to obtain a singlet

e.g. $\bar{R}R + \bar{Y}Y + \bar{B}B$ — we need the parallel transport factor to make sure that the antiquark paired with R is really $\bar{R}$ when carried back to x from y.]

After time T, we annihilate the pair with $M^\dagger(y,x)$ — the quark and antiquark are still at the same positions. It is convenient to fix the $A_0 = 0$ gauge, which removes the freedom to rotate the color basis as time passes — this ensures that the quark created by $\bar{q}_R$ will be annihilated by q_R (not a linear combination of q_R, q_Y, q_B).

Now — in Euclidean space, we have

$$\langle 0 | M^\dagger(T) M(0) | 0 \rangle$$

$$= \sum_n \langle 0 | M^\dagger(0) | n \rangle e^{-E_n T} \langle n | M(0) | 0 \rangle$$

where E_n is the energy of state $|n\rangle$. In the limit $T \to \infty$ (if there is an energy gap) this becomes dominated by the ground state in the presence of the source $|\bar{0}\rangle$

$$\xrightarrow[T\to\infty]{} |\langle \bar{0} | M(0) | 0 \rangle|^2 e^{-E(L)T},$$

where $E(L)$ is the minimal energy in the presence of the source. In $A_0 = 0$ gauge for static quarks, we have

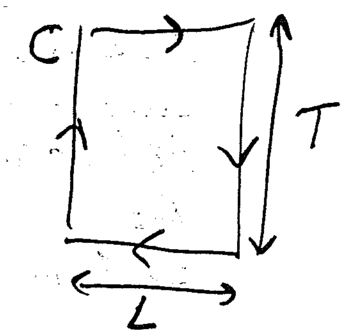
$$q^j(T, y)\, \bar{q}_i(0, y) = \delta^j{}_i,$$

so this quantity is

$$\langle 0 | [P \exp(iq \textstyle\int A)]^\dagger (T) [P \exp iq \int A](0) | 0 \rangle$$

which is just the $A_0 = 0$ gauge expression for

$$\langle 0|W(C)|0\rangle = \langle 0|\mathrm{tr}\,P\exp\left(ig\oint_C A\right)|0\rangle$$



– the expectation of the Wilson loop, where C is the rectangular loop shown.

In a confining phase, with

$$E(L) \xrightarrow[L\to\infty]{} KL \qquad (K = \text{string tension})$$

we anticipate that

$$\langle W(C)\rangle_{\text{Euclidean}} \xrightarrow[L,T\to\infty]{} \exp[-KLT]$$

$$= \exp[-K(\text{Area})]$$

where "Area" means the minimal surface area of a surface S with $\partial S = C$.

In a Higgs phase, we expect

$$E(L) \xrightarrow[L\to\infty]{} 2\mu = \text{constant},$$

or $$\langle W(C)\rangle_{\text{Euclid}} \xrightarrow[L,T\to\infty]{} \exp[-2\mu T]$$

$$\sim \exp[-\mu(\text{Perimeter})]$$

where "Perimeter" means arc length of C (and assuming $T \gg L$).

The asymptotic behavior ~~of~~ a large Wilson loop, then, provides a nonlocal order parameter that distinguishes confining behavior from Higgs behavior. (If we take Volume $\to \infty$ limit first, then $\langle W(C)\rangle$ is nonanalytic at the critical point.)

We inferred this behavior for timelike loops, but Euclidean invariance implies that the same criterion applies to spacelike loops, e.g. C lying in a time slice. What is the interpretation for a spacelike loop?

Quantizing in $A_0=0$ gauge, recall that the electric field E_i^a is conjugate to A_i^a (vacuum angle $\theta=0$) and can be represented as

$$E_i^a(x) = -i\frac{\delta}{\delta A_i^a(x)}$$

$$W(C) = \mathrm{tr}\, P\exp\left(ig\oint_C A_\mu dx^\mu\right)$$

$$\Rightarrow E_i^a(\vec{x})W(C) = W(C)E_i^a(\vec{x}) - i\frac{\delta}{\delta A_i^a(\vec{x})}W(C)$$

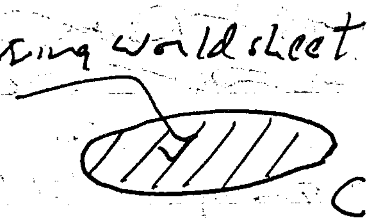
$$= W(C)E_i^a(\vec{x}) + g\,\mathrm{tr}P\left[\frac{\delta}{\delta A_i^a(\vec{x})}\left(\oint_C \vec{A}\cdot d\vec{\ell}\right)\right]\exp(ig\oint A)$$

This means that applying $W(C)$ to a state "shifts" the value of the electric field at points on the loop C. That is $W(C)$ acting at a fixed time creates a unit of electric flux (with strength g) along the loop C.

So W(C) is an operator that creates a string of electric flux, and

$$\langle 0 | W(C) | 0 \rangle$$

is the amplitude for that loop of flux to annihilate to the vacuum. In the Euclidean path integral picture, the history of minimal action in which a loop of string is created on C is one with the string world sheet sketched tautly across the minimum area surface bounded by C. This minimum action is



$$S \sim K \cdot \text{Area},$$

and we obtain

$$\langle W(C) \rangle \sim \exp(-K \cdot (\text{Area}));$$

that is what happens in the confinement phase. But in the Higgs phase, the electric flux is not locally stable — it can spread and dissipate. Since the decay of the string can be described locally along the string, we find

$$\langle W(C) \rangle \sim \exp(-\mu \cdot (\text{Perimeter}))$$

Magnetic Disorder

The Wilson loop criterion for confinement is useful. For one thing, it leads us to the insight that quark confinement can be explained if the vacuum of the pure Yang-Mills theory is magnetically disordered.

~~Recall that, if A in the vicinity of C is pure gauge, then for a particular~~

Consider a particular classical gauge field A that is "pure gauge" along the loop C. We know that then a fundamental representation Wilson loop in a $G = SU(N)/Z_N$ gauge theory has value

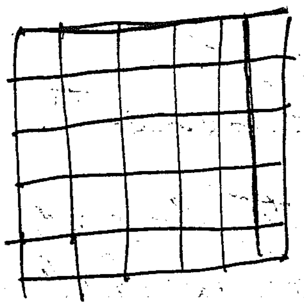
$$W(C) = e^{2\pi i K/N}\, tr(I) = N e^{2\pi i K/N},$$

where K (mod N) is the number of units of Z_N flux that thread the loop (pierce a surface bounded by the loop).

In the Higgs phase, magnetic flux is heavy. Z_N vortices arise rarely as fluctuations (in 2+1 dimensions), or virtual loops of Z_N string are heavily suppressed. But if $SU(N)$ is not Higgsed, then magnetic fluctuations are not so costly.

Confinement arises if magnetic fields have large fluctuations in the vacuum with weak long-distance correlations.
As a simple model, consider $G = SU(2)/Z_2$, so

$$\tfrac{1}{2} W(C) = 1 \quad \text{(even \# of vortices)}$$
$$\tfrac{1}{2} W(C) = -1 \quad \text{(odd \# of vortices)}$$



(surface bounding)
consider a large Wilson loop that can be divided into a large number N of cells, where the magnetic fluctuations in each cell can be regarded as uncorrelated with fluctuations in other cells (Roughly, L is the area divided by a^2, where a is a correlation length). Each cell contains

No vortex, Probability p
one vortex, Probability $(1-p)$

If we average over all configurations, the Wilson loop expectation value will be

$$\frac{1}{2}\langle W(C)\rangle = \langle (-1)^{n_v}\rangle$$

where n_v is the number of vortices enclosed by C

$$\langle (-1)^{n_v}\rangle = \sum_n \binom{N}{n} p^n (1-p)^{N-m} (-1)^n$$

which is just the binomial expansion of

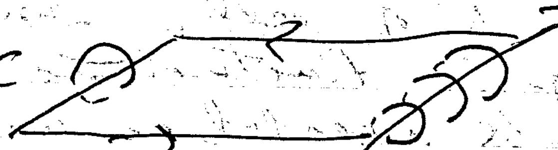
$$[1-p+(-p)]^N = (1-2p)^N$$

But if $N = \frac{\text{Area}}{a^2}$, this is

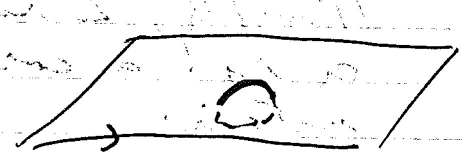
$$\langle W(C) \rangle \sim \exp[-K(\text{Area})]$$

where $K = \frac{-1}{a^2} \ln(1-2p)$. Magnetic disorder generates confinement.

In a Higgs phase, on the other hand, there is <u>magnetic</u> confinement.

C  Only short loops of string (or worldlines of vortices, in 2+1 dimensions) occur.

World lines deep inside C don't contribute to the total enclosed flux, and only cells near the edge of the surface bounded by C contribute to the fluctuations.



C

The number of cells N is proportional to the perimeter, and we find

$$\langle W(C) \rangle \sim \exp(-\text{Perimeter})$$

This is our first encounter with the concept of <u>electric-magnetic duality</u> in the theory of confinement. Magnetic confinement and electric confinement seem to be mutually exclusive, in the context of the simple model:

Lecture #28

Magnetic Disorder $\Longleftrightarrow$ Electric Confinement
Magnetic Confinement $\Longleftrightarrow$ Electric Disorder

Can we see that this connection applies more generally?

Abelian Higgs Model in 1+1 dimensions

Magnetic disorder can be analyzed semiclassically in some (1+1)- and (2+1)-dimensional gauge theories. We will discuss some examples.

In one spatial dimension, a gauge field is nondynamical. E.g. the component A_1 can be eliminated by choosing the gauge $A_1 = 0$ then $F_{01} = -\partial_1 A_0$. No time derivatives of A_0 appear in the action, so A_0 is a constrained variable determined on a time slice by the values of other fields, and by boundary conditions.

Nevertheless, the gauge field has important effects. There is a "Coulomb" phase that is actually confining. Two classical + and −ly charged sources generate a constant electric field, and an energy that diverges as the charges are separated.

$$q \bullet \xrightarrow{E} \bullet -q$$

If we couple A to a Higgs field, it is reasonable to expect a Higgs phase in which electric fields are screened, and charges are unconfined. This expectation is reasonable, but it is wrong. In fact there is no Higgs phase — a feature that we would miss if we confined our attention to perturbation theory, but can be seen if we study nonperturbative effects.

In (1+1)-dimensions, the vortices of the abelian Higgs model become instantons. Furthermore, there is a Θ parameter. The topological quantum number is the total flux in units of the flux quantum

$$\nu = \frac{e}{2\pi}\int d^2x\, F_{01}$$

$$= \frac{e}{4\pi}\int d^2x\, \epsilon_{\mu\nu}F^{\mu\nu},$$

i.e., the integral of the first Chern class.

In contrast to (3+1)-dimensional Yang-Mills theory, the (1+1)-dimensional Higgs model is not scale invariant — the instantons have a size $\sim (ev)^{-1}$, and there is no infrared problem to invalidate semiclassical reasoning at weak coupling.

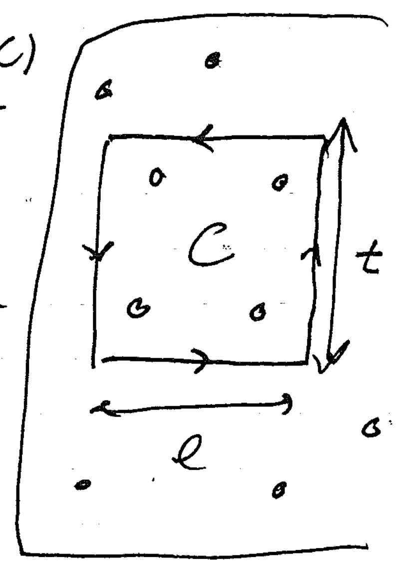
In the putative Higgs phase,

let's evaluate the expectation value of a large Wilson loop in the dilute instanton gas approximation. Consider a Wilson loop describing a source with charge q

$$W_q(C) = \exp\left(iq\oint A\right);$$

its expectation value in a θ-vacuum is

$$\langle W_q(C)\rangle_\theta = \frac{\int e^{-S}e^{i\nu\theta}\,W_q(C)}{\int e^{-S}e^{i\nu\theta}}$$



We have calculated the denominator previously. Up to a normalization that divides out of the expectation value, it is

$$\text{Denom} = \exp\left[\left(2Ke^{-S_0}\cos\theta\right)LT\right]$$

where S_0 is the instanton action, K is the determinant obtained from the Gaussian integral about the instanton background, and LT is the spacetime volume — we conclude that the vacuum energy is

$$\frac{E_0(\theta)}{L} = -2Ke^{-S_0}\cos\theta$$

In the numerator, there are contributions due to instantons that are inside C

and outside C. Since $S = S_{in} + S_{out}$ and $\nu = \nu_{in} + \nu_{out}$, the numerator factorizes as

$$\left(\int_{in} e^{-S} e^{i\nu\theta} W_q(C) \right) \cdot \left(\int_{out} e^{-S} e^{i\nu\theta} \right)$$

(the outside instantons have no effect on $W_q(C)$). The outside integral is

$$\exp\left[2Ke^{-S_0}\cos\theta \, (LT - \text{Area}) \right]$$

where "Area" is the area enclosed by C, the area from which outside instantons are excluded. For the inside instantons

$$W_q(C) = \exp\left(iq \oint_C A \right) = \exp\left(iq \frac{2\pi}{e} \nu_{in} \right);$$

this has the same effect as shifting the value of θ inside the loop to

$$\theta \to \theta + \frac{2\pi q}{e},$$

and the inside integral is

$$\exp\left[2Ke^{-S_0} \cos\left(\theta + \frac{2\pi q}{e}\right) \cdot \text{Area} \right]$$

Taking the ratio to find the expectation value:

$$\langle W_q(C) \rangle_\theta = \exp\left[2Ke^{-S_0} \left(\cos\left(\theta + \frac{2\pi q}{e}\right) - \cos\theta \right) \cdot \text{Area} \right]$$

we find a "string tension"

$$K_q(\theta) = 2Ke^{-S_0}\left[\cos\theta - \cos\left(\theta + \frac{2\pi q}{e}\right)\right]$$

Comments:

- Consider the case $\theta = 0$. We have

$$K = 2Ke^{-S_0}\left[1 - \cos\left(\frac{2\pi q}{e}\right)\right],$$

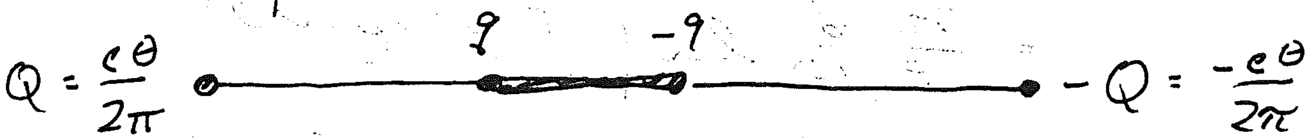
or $K > 0$ unless $q = e \cdot$ integer
the finding $K = 0$ for $q = e$ is not a surprise — the charged Higgs field can screen the source.
But for other values of q, we have confinement rather than the Higgs phenomenon. There is no evidence of a phase transition that separates the $v^2 > 0$ and $v^2 < 0$ regimes.

- However, the θ dependence of the string tension looks peculiar...
It is periodic in θ with period 2π as we expected (as well as being periodic in q/e with period 1), but for $\theta \neq 0$, there is a range in q centered at

$$q = -\frac{e\theta}{2\pi}$$

for which $K < 0$! What is going on?

It helps to have a physical interpretation of Θ. Indeed, Θ can be understood as a background charge at spatial infinity



In the Θ-vacuum, there is in effect a charge $Q = \frac{e\Theta}{2\pi}$ at the far left of the world, and opposite charge $Q = -\frac{e\theta}{2\pi}$ on the right. Perturbatively, we'd expect this charge to be screened by the Higgs condensate — but when nonperturbative corrections are included we find that it is only partially screened; far from the charge Q, the electric field approaches a non-zero constant.

We can calculate the expectation value of the electric field semiclassically. Recalling that

$$\Theta \nu = \frac{e\Theta}{2\pi} \int d^2x \, F_{01},$$

we see that (since $\langle F_{01}(x) \rangle$ is translation invariant)

$$\int d^2x \langle F_{01} \rangle_\theta = \frac{\int e^{-S} e^{i\nu\theta} \left(\int d^2x\, F_{01} \right)}{\int e^{-S} e^{i\nu\theta}}$$

$$= \frac{2\pi}{ie} \frac{d}{d\theta} \ln \left(\int e^{-S} e^{i\nu\theta} \right)$$

$$= \frac{2\pi}{ie} \frac{d}{d\theta} \ln \exp \left(2K e^{-S_0} \cos\theta \, LT \right)$$

$$= \frac{2\pi i}{e} \left(2K e^{-S_0} \sin\theta \right)(LT)$$

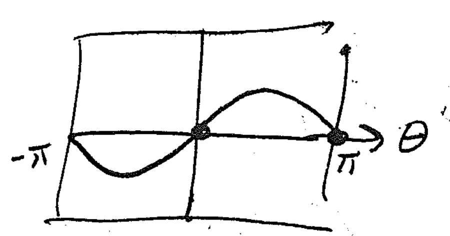
$$\Rightarrow \langle F_{01} \rangle_\theta = \frac{4\pi i}{e} K e^{-S_0} \sin\theta$$

– which continued to Lorentzian spacetime gives

$$\langle F_{01} \rangle_{\theta,\, \text{real time}} = \frac{4\pi}{e} K e^{-S_0} \sin\theta$$

– this expectation value is nonperturbatively small, but not zero, for $\theta/\pi \neq$ integer

($\theta = 0, \pi$ are the CP conserving values, for which $\langle F_{01} \rangle = 0$ follows from symmetries.)

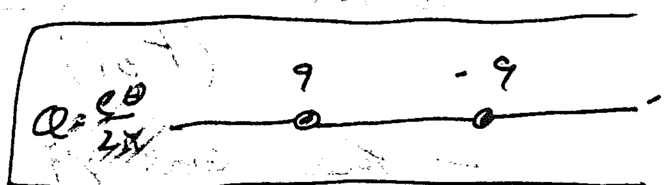


Our interpretation of θ as a background charge is reinforced by the observation that we would obtain the same result if we computed $\langle F_{01} \rangle$ <u>inside</u> a Wilson loop of infinite site (so we could appeal to translation invariance as above) — or, in other words, by the observation that inserting an ∞ Wilson loop is equivalent to shifting θ by $\theta \to \theta + \frac{2\pi q}{e}$.

Now the formula

$$K_q(\theta) = 2Ke^{-S_0}\left[\cos\theta - \cos\left(\theta + \frac{2\pi q}{e}\right)\right]$$

makes perfect sense. The vacuum energy is proportional to

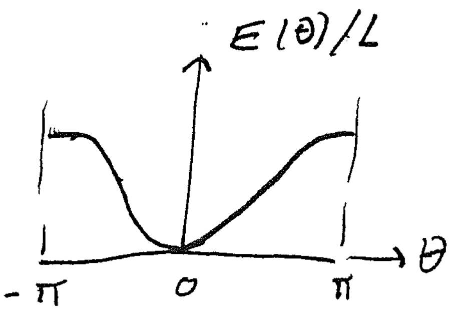


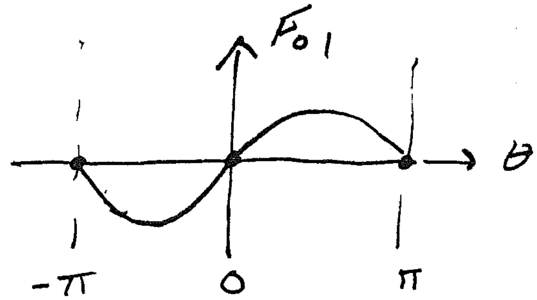
$$-\cos\theta = -\cos\left(2\pi\frac{Q}{e}\right)$$

In between the two test charges, the total background charge is $Q+q$ and the energy per unit length

is $\propto \cos\left(2\pi\,\frac{Q+q}{e}\right)$.

The change in vacuum energy density due to the test charges can be of either sign, so we can have + or − "string tensions" — either an attraction or repulsion between the charges.





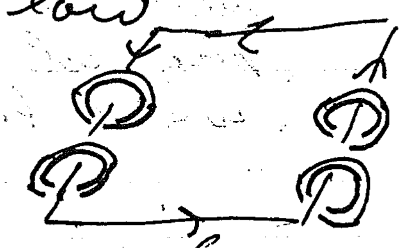
Why is it, though, that the energy density is maximal at $\theta = \pi$, where the average electric field vanishes? Although the mean value of F_{01} vanishes, there are large fluctuations that average to zero. The importance of these fluctuations is already apparent when we compare the energy density with

$$(F_{01})^2 \propto (e^{-S_0})^2$$

— the energy density $\propto e^{-S_0}$ is huge compared to the square of the average field strength. This is the behavior that arises if the electric field and vacuum energy look like those in the perturbative vacuum most of the time — but exponentially rarely there is a very large field fluctuation. At $\theta = 0, \pi$, the field fluctuations are as likely to be positive as negative, and their contribution to $\langle F_{01} \rangle$ cancels. But for other values of θ, the fluctuations are biased by the background charge.

Abelian Higgs Model in (2+1) dimensions

In the abelian Higgs model in 2+1 dimensions, there really is a Higgs phenomenon, and the vortices are stable particles in the Higgs phase. This phase is "magnetically ordered" and "electrically disordered" - Wilson loops obey the perimeter law



Long vortex world lines are suppressed in the path integral by

$$\exp\left(-M_{vortex} \cdot (\text{length})\right)$$

- therefore the fluctuations in the magnetic flux on a surface bounded by C are due to vortices near the edge of the surface — the world lines that <u>link</u> with C.

The other possible phase is the Coulomb phase. It is instructive to interpret the Coulomb-Higgs phase transition in terms of the realization, not of the local gauge symmetry, but of a <u>global</u> symmetry.

For this purpose, we will introduce a scalar field $\underline{\Phi}$ (not to be confused with the Higgs field ϕ) that can annihilate a vortex or create an antivortex. We will describe how this operator is constructed from two viewpoints: the canonical formalism, and the Euclidean path integral.

For the canonical formalism, we work in $A_0 = 0$ gauge, and construct a physical Hilbert space $\mathcal{H}_{phys}$ — wave functionals are

$$\psi[A] = \psi[A^{\Omega}]$$

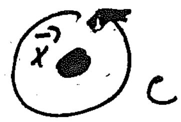
where $\Omega = \exp(-ie\,\omega(x))$ is a small gauge transformation — with $\omega(x)$ continuous on $\mathbb{R}^2$ and $\omega(x) = 0$ at spatial infinity.

In the Schrodinger picture, we define the action of our local vortex field $\underline{\Phi}(\vec{x})$ acting on a basis for the large Hilbert space, the basis of field eigenstates:

$$\hat{A}_i(\vec{x})\,|A_i(\vec{x})\rangle = A_i(\vec{x})\,|A_i(\vec{x})\rangle$$

on this basis we define

$$\underline{\Phi}(\vec{x})\,|A_i(\vec{y})\rangle = |A_i^{\Omega(\vec{x})}(\vec{y})\rangle.$$



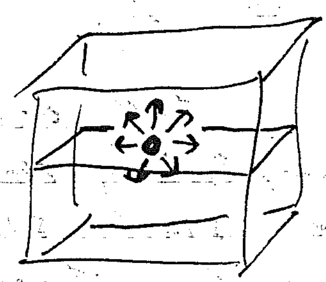
Here $\Omega(\vec{x})$ is a gauge that acts trivially at spatial infinity, and is continuous at each point in $\mathbb{R}^2$ <u>except</u> the point $\vec{x}$.

Around a loop C that encloses $\vec{x}$, $\Omega(\vec{x})$ has a winding number q (=1). It is a "singular gauge transformation" that in effect introduces a pointlike quantum of flux at the point $\vec{x}$.

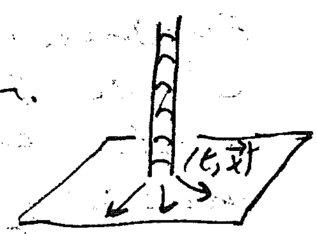
The action of $\Phi(\vec{x})$ on $\mathcal{H}_{phys}$ is completely determined by its behavior at the singularity — for two different gauge transformations Ω_1 and Ω_2 with the same singularity act the same way: $\Omega_1^{-1}\Omega_2$ is a continuous and small gauge transformation that is represented by I on $\mathcal{H}_{phys}$. Furthermore, acting on $\mathcal{H}_{phys}$, it is clear that $\Phi(\vec{x})$ is gauge invariant and commutes with other gauge invariant operators at spacelike separation — it is qualified for inclusion in the algebra of gauge-inv. local operators of the theory.

Furthermore, it is <u>not</u> redundant of include $\Phi(\vec{x})$ in the local field algebra — its action on $\mathcal{H}_{phys}$ cannot be duplicated by operators obtained by smearing gauge invariant polynomials in the gauge and Higgs field. E.g. if we compactify $\mathbb{R}^2$ to S^2, $\Phi(\vec{x})$ interpolates between sectors with different values of the topologically conserved magnetic flux.

Now the Euclidean path integral description of the (Heisenberg picture) operator $\Phi(t,\vec{x})$:



If $\Phi(t,\vec{x})$ is inserted, the path integral is restricted to configurations that have a Dirac (charge $-\frac{2\pi}{e}$) monopole singularity sitting at the point $(t,\vec{x})$ in Euclidean spacetime. You can see how this description connects with the canonical construction.



The monopole changes the magnetic flux:

$$\text{flux}\Big|_{\substack{\text{after}\\ \text{monopole}}} = \text{flux}\Big|_{\substack{\text{before}\\ \text{monopole}}} - \frac{2\pi}{e}$$

You can think of the monopole as the end of a dirac string that is created by the action of $\Phi(t,\vec{x})$. The string is invisible, but its end is not.

The vortex field $\underline{\Phi}$ is a complex scalar field. And since it changes the value of the topologically conserved magnetic flux, we can construct a unitary operator $U(\alpha)$ via

$$U(\alpha)\,|\text{flux}\rangle = \exp[-i\alpha\cdot(\text{flux})]\,|\text{flux}\rangle$$

(where "flux" denotes magnetic flux in units of $2\pi/e$), and since

$$\underline{\Phi}\,|\text{flux}\rangle = |(\text{flux})-1\rangle,$$

we have

$$U(\alpha)\,\underline{\Phi}\,U^{-1}(\alpha) = e^{i\alpha}\,\underline{\Phi}.$$

Furthermore, since flux is conserved $U(\alpha)$ is a <u>symmetry</u>, commuting with the Hamiltonian of the theory. We thus identify a <u>global</u> $U(1)$ symmetry (<u>not</u> to be confused with the gauge $U(1)$!) under which $\underline{\Phi}$ transforms as

$$\underline{\Phi} \to e^{i\alpha}\,\underline{\Phi}$$

We call this the "Topological $U(1)$ symmetry" of the $(2+1)$-dimensional abelian Higgs model.

How is the topological symmetry realized? First consider the Coulomb phase, with a massless photon. Suppose we introduce an IR cutoff by compactifying on a two-sphere of finite volume. We can probe whether SSB of the global $U(1)$ occurs by investigating whether superselection sectors appear in the limit of infinite volume.

In finite volume, the theory has a ground state $|n\rangle$ in the sector with magnetic flux $n\cdot(2\pi/e)$, and these are split in energy - the unique ground state is in the $n=0$ sector.

But the energy is $\int \frac{1}{2}B^2 \sim \frac{1}{2}\text{Area}\cdot\left(\frac{\text{Flux}}{\text{Area}}\right)^2 = \frac{1}{2}\frac{(\text{Flux})^2}{\text{Area}}$

which approaches 0 as Area $\to \infty$, and the vacuum becomes infinitely degenerate.

The vortex operator Φ is not diagonal in the $|n\rangle$ basis, as

$$\langle m|\Phi|n\rangle = c\,\delta_{m,n-1},$$

so for the purpose of studying the realization of the $U(1)$, it is preferable to use the basis

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} e^{in\theta}|n\rangle$$

such that

$\langle\theta'|\theta\rangle = \delta(\theta'-\theta)$ and $\langle\theta'|\Phi|\theta\rangle = c\,e^{i\theta}\delta(\theta-\theta')$

The topological symmetry acts according to

$$U(\alpha)|n\rangle = e^{-in\alpha}|n\rangle,$$

and hence $U(\alpha)|\theta\rangle = |\theta - \alpha\rangle$
— the symmetry is spontaneously broken.

If a $U(1)$ global symmetry is spontaneously broken, then there must be a Goldstone boson, a massless spin-0 particle. In (2+1)-dimensional electrodynamics, the Goldstone boson is the photon. Note that in 2 spatial dimensions there is no helicity (no rotations that preserve the momentum) and therefore massless particles (including the photon) have no spin.

[The perturbative counting of degrees of freedom matches between Higgs and Coulomb phases: 2 scalars and a photon in Coulomb phase, one scalar and a two component vector in Higgs phase.]

On the other hand, in the Higgs phase, the $|n\rangle$ sectors remain nondegenerate as volume $\to \infty$, since the vortex has nonzero mass, and the ground state is the nondegenerate $|n=0\rangle$ state. The topological $U(1)$ is manifest, and $\Phi(\vec{x})$ acting on the vacuum creates an antivortex. There is a gap, and no Goldstone boson.

Thus we learn that the Higgs/Coulomb transition in 2+1 dimensions reflects the change in realization of a global symmetry:

Spontaneously Broken Topological $U(1)$ $\Longleftrightarrow$ Coulomb phase (massless photon as Goldstone boson)

Manifest topological $U(1)$ $\Longleftrightarrow$ Higgs phase (conserved vortex number

Lecture #29

Confinement and Instantons in (2+1) dimensions

In (2+1) dimensions, the abelian Higgs model does not have instantons. But we can embed it in a theory that does admit instanton solutions.

Consider (as already discussed several times) the hierarchy of symmetry breakdown

$$SU(2) \xrightarrow{v_1} U(1) \rightarrow ??$$

This theory has 't Hooft-Polyakov monopoles that carry flux $4\pi/e$, and these become instantons in (2+1) dimensions

How do the instantons affect the topological symmetry?

In fact they break the symmetry intrinsically (not spontaneously). The monopole is a quantum tunneling process that changes the magnetic flux by 2 units (and an antimonopole by (-2) units). "Integrating out" monopoles and antimonopoles generates a term in the effective Lagrangian that annihilates or creates 2 vortices, with an exponentially small coefficient

$$\mathcal{L}' \sim C e^{-S_0} \left(\Phi^2 + (\Phi^\dagger)^2 \right)$$

($S_0 \sim 4\pi v/e$ is the monopole action). Because of the intrinsic breaking, the massless Goldstone boson becomes a very light pseudo-Goldstone boson. In other words, the photon gets a mass, and the theory is no longer in the Coulomb phase.

In a language more appropriate to electrodynamics (in 3 Euclidean dimensions), there is Debye screening due to the monopole-antimonopole plasma — correlations decay exponentially and there is a mass gap.

Indeed, instead of a Coulomb phase, the monopole instantons generate a confinement phase. Like the instantons of the (1+1)-dimensional model, they induce magnetic disorder, and hence electric confinement. Because the Debye screening, the widely separated monopoles and antimonopoles become uncorrelated (or have exponentially weak correlations) and suffice to magnetically disorder the vacuum.

We can infer that $\langle W(C)\rangle$ exhibits area law behavior by considering the behavior of a correlation function

$$\langle W(C)\,\Phi(x)\rangle$$

of a Wilson loop and a vortex operator. Let's imagine that the monopoles have charge $2\pi/e$ (Dirac charge) and that $\Phi(x)$ annihilates flux $2\pi/e$ (flux quantum). To keep the discussion interesting, we take $W(C)$ to be the charge e Wilson loop

$$W(C) = \exp\left(ie\int A\right).$$

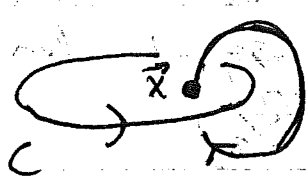
Then whether the monopole plasma really suffices to drive confinement is a somewhat subtle question, since the Wilson loop is unable to detect the flux $2\pi/e$ that is consumed or created in a tunneling event.

We'll consider pure electrodynamics (no charged matter) but with the effects of monopole instantons included in the effective Lagrangian. Actually, all we'll need to know is that we're in a phase in which the topological $U(1)$ symmetry is both spontaneously broken, so that

$$\langle \Phi \rangle = v e^{i\sigma} \neq 0,$$

and intrinsically broken, so that the phase σ has a preferred value, e.g. $\sigma = 0$.

First ignore the intrinsic breaking (the Coulomb phase of electrodynamics).



Suppose the loop C is far from the point $\vec{x}$

$$\langle W(C)\, \Phi(\vec{x}) \rangle$$

and consider as the point $\vec{x}$ winds (in 3d Euclidean space) about the loop C

The correlation factorizes as

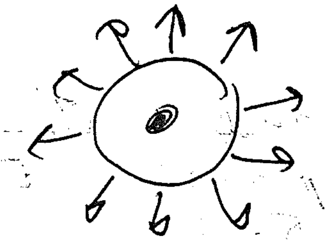
$$\langle W(C)\, \Phi(\vec{x}) \rangle \sim \langle W(C) \rangle \langle \Phi(\vec{x}) \rangle,$$

but we can also see that the phase of the correlation function rotates by 2π as $\vec{x}$ winds — as in effect we are pulling one dirac charge through the loop C.

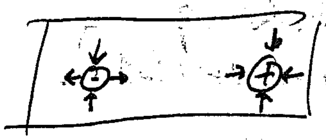
The interpretation, in the Coulomb

phase, is that $W(C)$ creates a loop of "global string" along the loop C
i.e. the Goldstone phase of the order parameter Φ winds by 2π around the string

A global vortex has a logarithmically divergent energy



$$E \approx \frac{1}{2} v^2 \int d^2x \left(\frac{1}{r}\frac{d\sigma}{d\theta}\right)^2 = \frac{1}{2} v^2 \, 2\pi \int \frac{dr}{r} \sim \pi v^2 \log R$$



And a vortex pair with separation R has energy of order

$$E(R) \sim \pi v^2 \log(R/a)$$

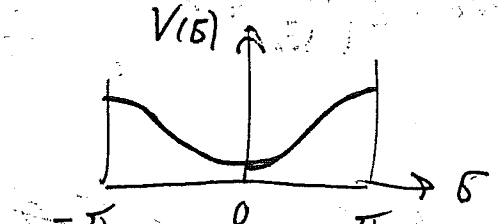
where a is the thickness of the string core.
This logarithmic energy is just a dual description of the Coulomb energy (in 2 dimensions)

$$E(R) = \int_a^R d^2x \left(\frac{e}{2\pi r}\right)^2 \sim \frac{e^2}{2\pi} \log R/a$$

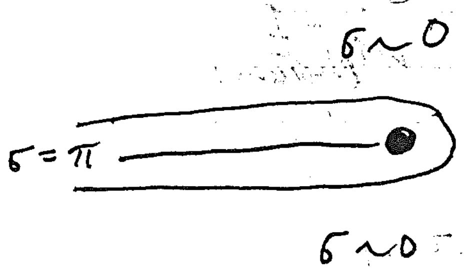
So the Wilson loop, which describes a charged source in the language of electrodynamics, actually describes a "vortex" of the Φ field in the vortex field language. We also see that vev of Φ is $\langle\Phi\rangle \sim e$, where e is the gauge coupling

[Careful. In terms of electrodynamics, Φ destroys a vortex – i.e. a quantum of flux. It is a vortex in the Φ field that describes an electric charge.]

Now what happens when we include the effects of intrinsic breaking of the topological $U(1)$ symmetry?



Now the phase σ of $\Phi = v e^{i\sigma}$ has an energetically preferred vacuum value which minimizes an exponentially small potential $V(\sigma) \propto e^{-S_0}$



As a result the global string becomes the boundary of a domain wall across which σ winds by 2π

This wall is like the kink studied earlier, with

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} v^2 \left(\frac{d\sigma}{dx}\right)^2 + V(\sigma)\right]$$

and $K \quad = v\int_0^{2\pi} d\sigma \sqrt{2V(\sigma)} = O(v\Lambda^{3/2})$

where the energy per unit area of the barrier is of order Λ^3. We can estimate

The thickness of the wall as

$$K \sim \Lambda^3 \cdot \text{thickness},$$

or Thickness $= \dfrac{K}{\Lambda^3} = \dfrac{v}{\Lambda^{3/2}} \sim m_{GB}^{-1}$

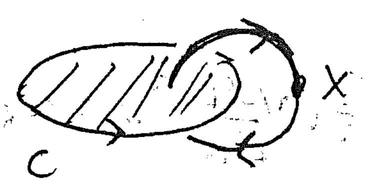
In fact this is just $\sim m_{GB}^{-1}$, where $m_{GB} \sim$ the pseudo-Goldstone mass:

$$m_{GB}^2 \sim \frac{\Lambda^3}{v^2}$$

What we have found then, is that the loop C becomes the boundary of a "domain wall" — which is just the confining electric flux tube!

Since $m_{GB} = m_{photon}$ and $v \sim e$, the thickness of the tube is m_{photon}^{-1}, and the string tension is

$$K = v\Lambda^{3/2} = v^2 m_{GB} \rightarrow e^2 m_\gamma$$

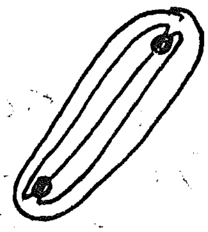


The winding of the phase in

$$\langle W(C)\, \Phi(x) \rangle$$

occurs when the point x crosses the world sheet of the electric flux tube, that is, the domain wall.

What happens in the Higgs phase?
Monopoles become <u>confined</u>



— Monopoles and antimonopoles are joined in Euclidean spacetime by locally stable magnetic flux tubes.

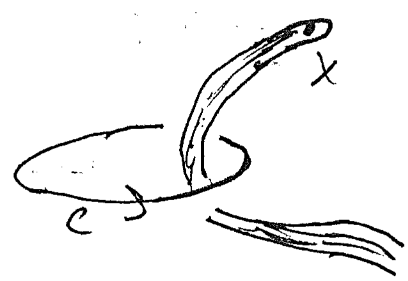
Thus the positions of monopoles and antimonopole become very strongly correlated. Pairs with large separation L are suppressed by

$$e^{-S} \sim \exp[-\mu L], \quad \mu = \text{magnetic string tension}$$

Monopoles are quantum tunneling events that create or destroy a magnetic vortex, and vortex number is no longer a conserved quantity.

Since monopoles are correlated, the vacuum is magnetically ordered — there is no electric confinement.

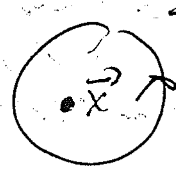
In the dual vortex operator language, the (approximate) topological $U(1)$ symmetry is not spontaneously broken in the Higgs phase; there is no light pseudo-Goldstone boson, and $W(C)$ does not create a loop of global string.
Rather, the phase of $\langle W(C)\, \Phi(x) \rangle$ winds when the magnetic flux tube ending at x crosses the loop C.



Higgs/Confinement in (2+1)-dimensional Yang-Mills Theory

Now consider a gauge theory with $G = SU(N)/Z_N$ (perhaps including adjoint Higgs fields). Now there is a topologically conserved Z_N magnetic flux in a Higgs phase. Just as before, we can construct a vortex field $\Phi(x)$, which now destroys a unit of Z_N flux — it can annihilate a vortex or create an antivortex.

In the canonical formulation, $\Phi(x)$ is a "singular gauge transformation" with



winding -1 on a closed path C that encloses x once.

In the Euclidean path integral, an insertion of $\Phi(x)$ restricts the sum over histories to those with a Z_N antimonopole singularity at the point x.

The theory has a topological Z_N symmetry. Unitary operators $\{U_n, n=0,\dots,N-1\}$, representing Z_N, act on the flux sectors $\{|m\rangle, m=0,\dots,N-1\}$ according to

$$U_n|m\rangle = \exp\left(-2\pi i \frac{nm}{N}\right)|m\rangle$$

and the vortex field Φ with the property

$$\Phi : |m\rangle \to |m-1\rangle$$

satisfies
$$U_n \Phi U_n^{-1} = e^{2\pi i n/N} \Phi .$$

We say that the gauge theory has a topological global Z_N symmetry, under which the vortex field transforms.

To summarize: in the $U(1)$ gauge theory, the topological magnetic flux has integer values, so the topological symmetry is $U(1)$, a continuous symmetry. In the $SU(N)/Z_N$ gauge theory, magnetic flux takes values in Z_N, and the topological symmetry is Z_N

The realization of the symmetry has quite different implications in the two cases. If the $U(1)$ global symmetry is spontaneously broken, there is a massless Goldstone boson, the photon. That's the Coulomb phase. If the Z_K symmetry is spontaneously broken, there is no Goldstone boson, but there is a stable domain wall. The wall is an electric flux tube — this is the confinement phase.

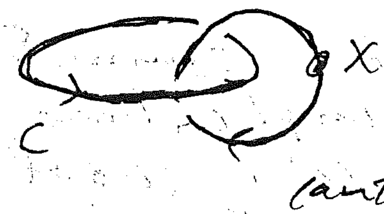
For $G=U(1)$, we need intrinsic breaking of topological symmetry (magnetic monopoles) to get stable domain walls and hence electric confinement. For $G=SU(N)/Z_N$, spontaneous breaking of topological symmetry does the job.

The consequences of the Z_N symmetry can again be appreciated if we consider the behavior of the correlation function

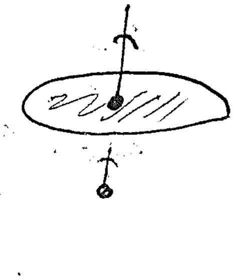
$$\langle W(C)\,\Phi(x)\rangle$$

where $W(C)$ is the fundamental irrep Wilson loop in $SU(N)$.

This correlation function is actually multivalued. If we carry



x around a closed path that winds once about C, we are pulling a Z_N (anti-)monopole through the Wilson loop, advancing its phase by $e^{-2\pi i/N}$. The correlation function has N sheets. We can confine attention to one sheet by introducing a jump by $e^{-2\pi i/N}$ when x crosses a surface bounded



by C.

Now — if x is always far from C, the correlation will factorize

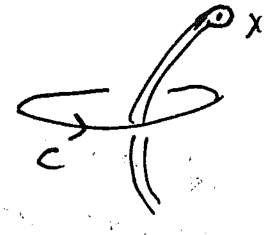
$$\langle W(C)\,\Phi(x)\rangle \sim \langle W(C)\rangle\langle\Phi(x)\rangle$$

And if we stick with our convention of restricting the correlation function to one sheet, we can regard it as single valued.

Now $\langle\Phi(x)\rangle \neq 0$ signifies that the vacuum is subject to strong magnetic fluctuations, and that the Z_N symmetry is spontaneously broken. In that case, the kinematic jump in the correlation as x winds must be compensated by a dynamical jump in the phase by $e^{2\pi i/N}$ that compensates. If this jump occurs on a sheet bounded by C, we conclude that the sheet is the world sheet of the electric string created by W(C).

The point is that the multivaluedness signifies that W(C) can create a "Z_N global string" around which the phase of Φ winds by $e^{2\pi i/N}$. Spontaneously breaking of Z_N means that $\langle\Phi\rangle$ has a preferred value, so its jump has to occur on a domain wall — the electric flux tube.

On the other hand, in the Higgs phase $\langle \Phi \rangle = 0$, and Φ acting on the vacuum produces a stable particle, the Z_N antivortex. So x is the end of a Z_N magnetic flux tube, and the correlator jumps as the magnetic string crosses C.



It's natural to wonder what happens if, as in our discussion of $G = U(1)$, we introduce monopole instantons — E.g. via an underlying Higgs breakdown

$$SU(3) \to SO(3) = SU(2)/Z_2$$

Now we can have Z_2 monopoles in our (2+1)-dimensional gauge theory with $G = SU(2)/Z_2$. Then Z_2 topological sym is intrinsically broken since quantum tunneling can change the Z_2 vortex number.

However, there is nothing much interesting to say about the implications of the intrinsic breaking for <u>confinement</u>: It doesn't make sense to introduce fundamental irrep Wilson loops if we also admit magnetic monopoles that don't respect the Dirac quantization condition with fundamental irrep electric charges. (The Wilson loops can see the Dirac string!)

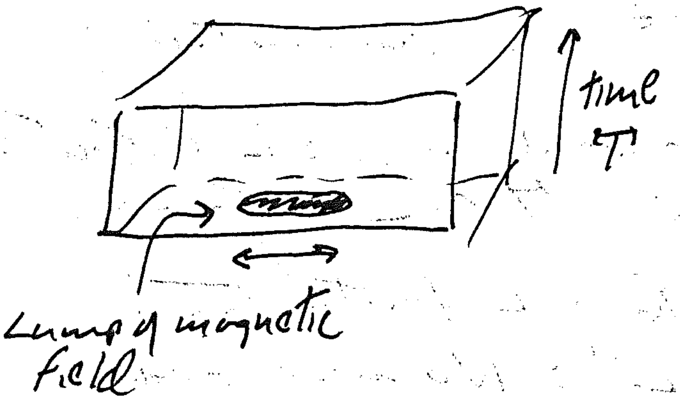
E.g. a Z_N fluxon has an Aharonov-Bohm interaction with a fundamental electric charge — but the monopole allows the fluxon to <u>disappear</u> due to quantum tunneling! We may include the monopoles or include the sources, but it does not make sense to do both!

Our arguments so far relate the Higgs/Confinement issue in (2+1)-dim Yang-Mills to the realization of a global symmetry, but don't pin down whether the Higgs or Confinement option is realized in, say, the pure YM theory without matter.

But we do have a very important hint: that the key difference between nonabelian and abelian gauge theories is that the topological magnetic flux takes values in Z_N rather than Z. Let's try to understand the implications better.

Mass Gap

Of course pure electrodynamics (no matter) is a free field theory, and we know everything about it. But it is instructive to see how the "Gaplessness" of the theory can be understood in the Euclidean path integral formalism



In a theory with a mass gap, field fluctuations decay quickly in Euclidean time, like e^{-mT}

But in a theory without a gap, there are fluctuations with wavelength L that have energy $\sim 1/L$, and so require a time of order L to collapse.

For example -- consider an initial configuration with a magnetic field that is coherent over a region of linear size L, with B large compared to typical root-mean-square vacuum fluctuations averaged over the region. What is the action of a field history in which the field disappears in time T?

In order of magnitude $B \sim A/L$, where A is vector potential and $E = \dot{A} \sim A/T$

$$S_E \sim \underbrace{A^2 L^2 T}_{\text{volume}} \left(\frac{1}{T^2} + \frac{1}{L^2} \right)$$

"kinetic energy" "potential energy"

or $S_E \sim A^2 \left(\frac{L^2}{T} + T \right)$

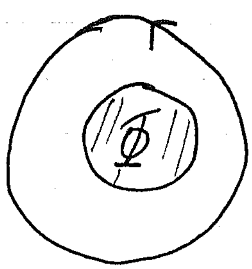
We minimize this by choosing $T \sim L$

(The fluctuation really dissipates by spreading out.)

the amplitude $\int_{B.C.} e^{-S_E}$ is dominated by configurations in which a size L fluctuation disappears in a time $\sim L$. Thus, a wavelength L fluctuation has energy $\sim L^{-1}$: No mass gap!

Why does it take so long? The trouble is that the relaxation of the long wavelength fluctuation requires a process that is coherent over a region of size L, and if that happened rapidly, there would be a large cost in kinetic energy.

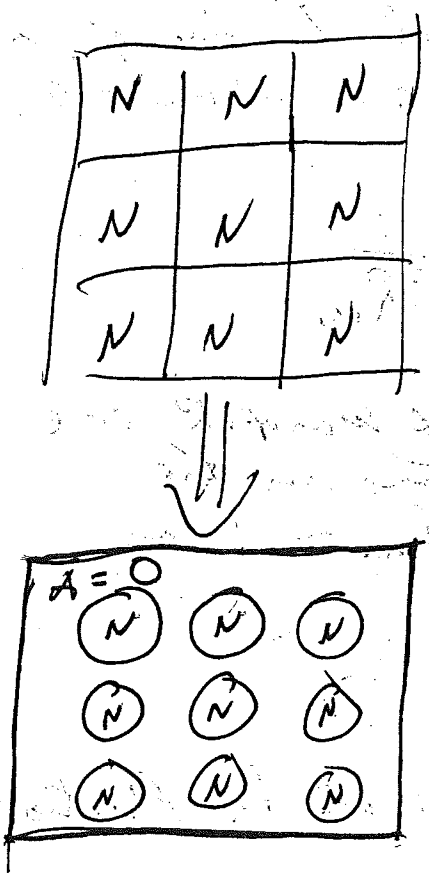
Why can't the flux just relax independently in many small regions? Because flux is locally conserved... actually the flux does not really disappear; it spreads out. From just the kinematic constraint $dF = d^2A = 0$,



we have $\text{curl}\, E = -\frac{\partial}{\partial t} B \Rightarrow \oint E \cdot dl = -\dot{\Phi}$

so $E \sim \frac{1}{2\pi R} \dot{\Phi}$, and kinetic energy $\frac{1}{2}\int E^2 \sim \dot{\Phi}^2 \log R$ would be IR divergent if flux really disappeared. The spreading of the flux takes a while...

Yang-Mills theory is completely different because the topology of the gauge group is different — in particular because N units of flux is equivalent to no flux



Imagine dividing region into cells, each containing N units of Z_N flux, i.e.

$$P \exp\left(i \oint A\right) = I$$

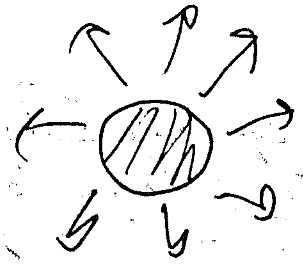
around boundary of cell

We can locally deform this to the configuration shown, $A = 0$ outside of non-intersecting disks that each enclose N units of flux.

And -- the configuration in each disk can be smoothly deformed to $A = 0$.

We can divide and conquer! Each region with N units of Z_N flux can relax independently, and the time for the fluctuation to relax is independent of its size L $\Rightarrow$ there is a mass gap!

How large is the gap?



viewed in Euclidean spacetime, the relation of the field is a *magnetic monopole* with N units of Z_N charge. Its topological Z_N charge is trivial, so it can have a nonsingular core, and its action is

$$S_{\text{monopole}} \sim \frac{1}{e^2 R_{\text{core}}}$$

where R_{core} is the size of the monopole core. The typical configurations in the path integral have $S_{\text{mon}} \sim 1$, or

$$R_{\text{core}} \sim \frac{1}{e^2}$$

– this is then the time scale for fluctuations to decay, or

$$\text{mass gap} \sim e^2.$$

Of course – this follows from dimensional analysis once we conclude gap $\neq 0$; e^2 is the only scale in the theory

Confinement

And why doesn't the abelian theory confine? In a region with $T \sim L$

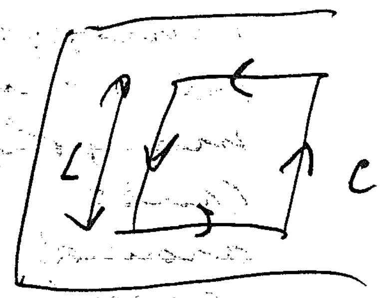
$$S_E \sim A^2 L \sim B^2 L^3,$$

so the typical vacuum fluctuations of B in a region of size L that dominate

$$\int e^{-S_E}$$

have

$$\langle B^2 \rangle^{\frac{1}{2}} \sim L^{-3/2}$$



and therefore the typical flux due to a size L coherent configuration through a loop C with linear size L is

$$\Delta\Phi \sim \langle B^2 \rangle^{\frac{1}{2}} L^2 \sim L^{1/2}$$

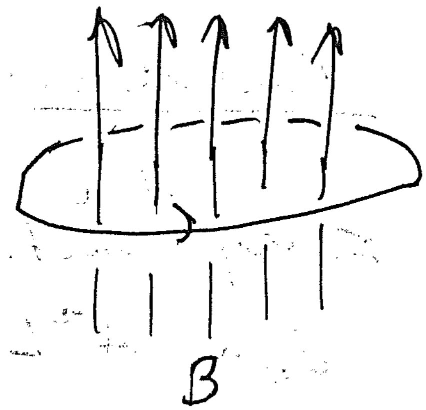
Averaging over these fluctuations gives a contribution to the Wilson loop

$$\langle W(C) \rangle = \langle e^{i\Phi} \rangle \sim \int d\Phi \, e^{i\Phi} e^{-\frac{1}{2}\Phi^2/\Delta\Phi^2}$$

$$\sim e^{-\Delta\Phi^2} \sim e^{-L}$$

— perimeter law behavior.

Furthermore, as already noted, shorter wavelength fluctuations don't help, because they are too



strongly correlated.
(the shorter wavelength fluctuations make the field lines wiggle, but they don't have much effect on the total flux through the loop.
(Large change in flux $\Rightarrow$ big electric field $\rightarrow$ large kinetic energy)

But in the YM theory, there are weakly correlated fluctuations that contribute to the value of $W(C)$ — these are the charge N-monopoles, which gradually change the flux by N units and so (as a monopole passes through the loop), cause the phase of $W(C)$ to wind around a closed path. In this respect, they are like our monopole instantons of the $U(1)$ theory.

The typical size of the monopoles is e^{-2}, which is thus the size of the weakly correlated fluctuation regions.

The number of fluctuation regions contributing to $W(C)$ is of order $e^4 \cdot \text{Area}$, and we find string tension

$$K \sim e^4 \sim e^2 M_{gap}$$

— like we found on p. 389, and what follows from dimensional analysis provided $K \neq 0$.