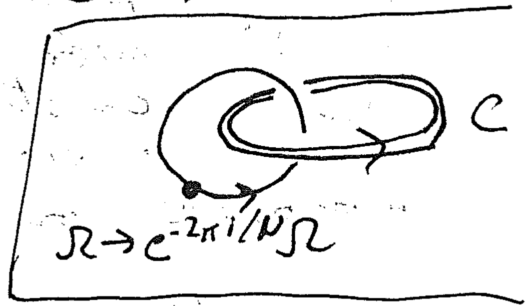
Yang-Mills in (3+1) dimensions

To what extent can our insights into confinement in (2+1) dimensions be extended to (3+1) dimensions?

In 3+1 dimensions, there is no conserved vortex number in the Higgs phase, and so strictly speaking no Z_N topological symmetry (at least at zero temperature). Still, some of the preceding discussion can be generalized.

Although the vortex is not a stable particle, there is a locally stable magnetic flux tube in the Higgs phase, and an "'t Hooft loop" operator $B(C)$ - the "'t Hooft loop" - that can create or destroy strings of flux.

We can construct the 't Hooft loop operator $B(C)$ the same way we constructed the vortex operator $\Phi(x)$. In the canonical language, $B(C)$ acting on a time slice in $A^0 = 0$ gauge is a gauge transformation, trivial at spatial infinity, singular on C, and with winding by $e^{-2\pi i/N} \in Z_N$ on a closed path that links once with C in the sense defined by the right-hand rule.



Therefore, $B(C)$ creates an infinitesimally thick line of Z_N magnetic (anti) flux.
As before, the action of $B(C)$ on physical states depends only on the nature of the singularity at C of the gauge transformation $\Omega(C)$:

$$B(C)\,|A_i(\vec{x})\rangle = |A_i^{\Omega(C)}(\vec{x})\rangle$$

We can see that

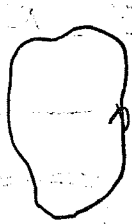
$$[B(C), B(C')] = 0$$

if the loops C and C' don't intersect, and that

$$[B(C), \mathcal{O}(R)] = 0$$

if $\mathcal{O}$ is a gauge invariant operator smeared in a region R that doesn't intersect C.

In the Euclidean path integral description,



C

an insertion of $B(C)$ in the path integral restricts the sum over histories to those constrained so that there is a world line of a singular Z_N monopole on the closed loop C.

Hence — $W(C)$ for spacelike C creates a closed line of electric Z_N flux, and for timelike C represents an insertion of a classical source carrying Z_N electric charge on worldline C.

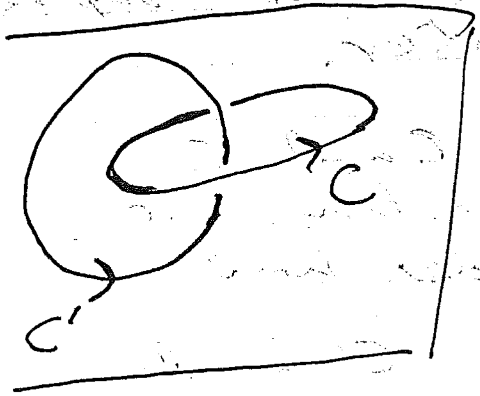
And $B(C)$ for spacelike C creates a closed line of Z_N magnetic flux, while for timelike C it represents an insertion of a classical Z_N monopole source.

Thus the Wilson loop $W(C)$ and 't Hooft loop $B(C)$ are really on the same footing, and are related by an electric-magnetic duality transformation that interchanges electric and magnetic fields.

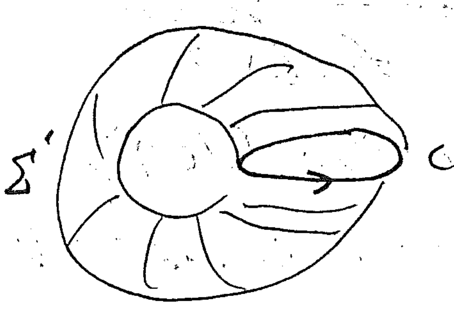
As in 2+1 dimensions, it is instructive to consider the correlation function

$$\langle W(C)\, B(C') \rangle,$$

which as before is multivalued.



There is an Aharonov-Bohm interaction between a Z_N charge and a Z_N magnetic flux tube, or between a Z_N magnetic monopole and a Z_N electric flux tube.



Imagine deforming C' so that it sweeps out a closed world sheet Σ'. In 4-dim Euclidean space, the surface Σ' has a linking number with the loop C;

this is the (signed) number of times that the string C' winds around the charge on the worldline C. The deformation that starts and ends with the loop C', then, has the effect on the correlation function:

$$\langle W(C)\, B(C')\rangle \rightarrow e^{2\pi i K/N} \langle W(C)\, B(C')\rangle$$

where K is the linking number of Σ' with C. Thus the correlation function is multivalued. Alternatively and equivalently, we can imagine carrying C around a closed surface Σ that links with C' — then an electric string winds around a magnetic monopole, and again there is a Aharonov-Bohm phase that depends on the linking number.

Now imagine that C and C' are at all times far from one another, and consider how $\langle W(C)\, B(C')\rangle$ behaves as C' winds around Σ'. The fluctuations near C and C' should look like fluctuations in the vacuum, so we expect the correlator to factorize

$$\langle W(C) B(C')\rangle \sim e^{i\alpha(\theta)} \langle W(C)\rangle \langle B(C')\rangle$$

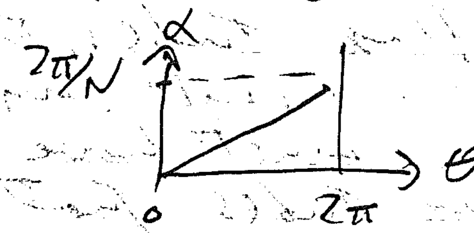
— up to a phase that may vary as C' is distorted;

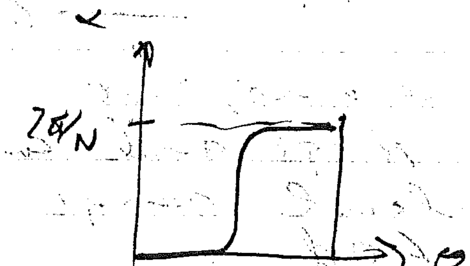
here $\theta \in [0, 2\pi)$ parametrizes a one-parameter family of loops, beginning and ending with C'. We know that α must increase by $2\pi/N$,

$$\alpha(\theta=2\pi) = \alpha(\theta=0) + 2\pi/N \pmod{2\pi}$$

if C' links once with C.

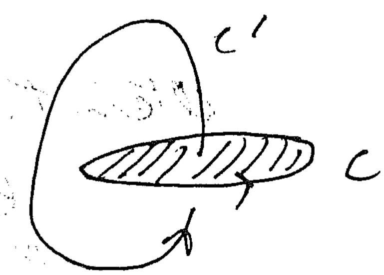
If the theory has massless excitations, then conceivably $\alpha(\theta)$ could increase smoothly. But this is not possible if the theory has a mass gap, and C, C' are far apart compared to the correlation length.





What must happen instead is that $\alpha(\theta)$ jumps suddenly. A sudden jump occurs if there is a physically observable sheet stretched across either C or C' —

In the former case, the jump occurs as C' crosses the sheet, in the latter case when C crosses the sheet.



If there is a sheet stretched across C, it has action $\propto$ Area of the sheet, and so we find

$$\langle W(C)\rangle \sim \exp[-K\cdot(\text{Area})]$$
– electric confinement

If there is a sheet stretched across C', then

$$\langle B(C)\rangle \sim \exp[-\mu\cdot(\text{Area})]$$
– magnetic confinement

We conclude, for purely topological reasons, that if the theory doesn't have stable magnetic flux tubes that confine monopoles (is not "magnetically ordered") then it must have stable electric flux tubes that confine quarks (is confining)

In 2+1 dimensions, we could distinguish Higgs and confinement phases with the local gauge invariant vortex field

$$\langle \Phi(x)\rangle = 0 \Rightarrow \text{Higgs}$$
$$\langle \Phi(x)\rangle \neq 0 \Rightarrow \text{Confinement}$$

Alternatively, we could consider

$$\lim_{|x-y|\to\infty} \langle \Phi^\dagger(x)\Phi(y)\rangle \begin{cases} =0 & \text{Higgs} \\ \neq 0 & \text{Confinement} \end{cases}$$

Although we don't have a local order parameter in 3+1 dimensions, the 't Hooft loop provides a similar criterion:

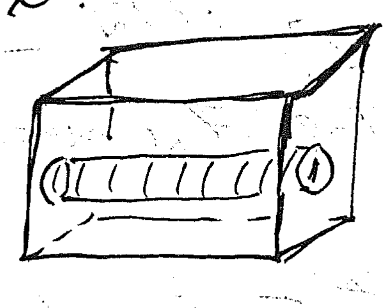
$$\lim \langle B(C)\rangle \sim \begin{cases} \exp(-\mu\cdot \text{Area}) \Rightarrow \text{Higgs} \\ \exp(-m\cdot \text{Perimeter}) \Rightarrow \text{confinement} \end{cases}$$

— in fact, though, we have not yet excluded the possibility of Higgs <u>and</u> confinement — i.e. a phase with simultaneous Area-law behavior for both $B(C)$ and $A(C)$.

Electric-Magnetic Duality

We can go further by exploiting more fully the duality symmetry relating $W(C)$ and $B(C)$.

We can probe for electric and magnetic confinement by studying the energetics of gauge fields on a 3-torus — i.e. a periodically identified box.



There are topological electric and magnetic fluxes associated with each fundamental nontrivial cycle of the torus. If the ground state energy in the sector with, say, electric flux running around the box in the $\hat{e}_3$ direction grows linearly with the size of the box, then there are stable electric flux tubes. Similarly for magnetic flux. E.g. if flux can spread,

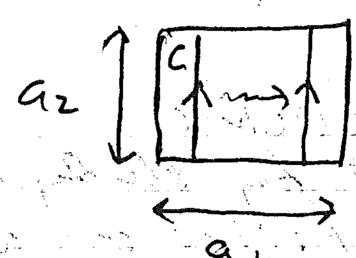
$$\text{Energy} \sim \left(\frac{\Phi}{a^2}\right)^2 a^3 \sim \frac{\Phi^2}{a} \to 0 \text{ as } a \to \infty$$

(where a is linear size of box)

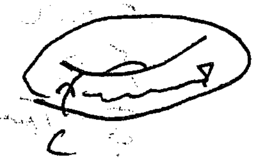
If flux is confined

$$\text{Energy} \sim K a \to \infty \text{ as } a \to \infty.$$

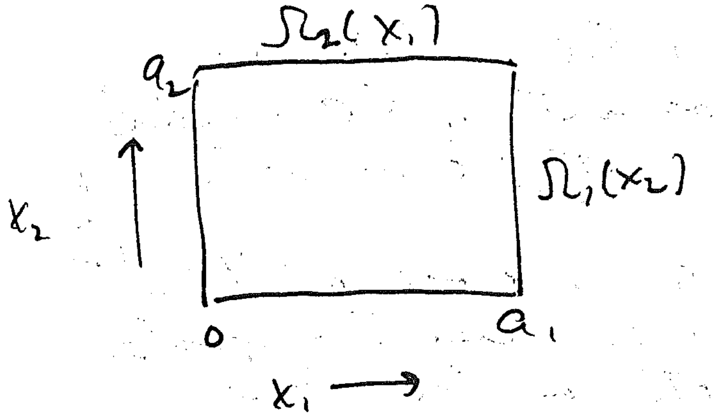
We have discussed magnetic flux on a 2-torus before. For $G = SU(N)/\mathbb{Z}_N$, consider a fundamental irrep Wilson loop C wrapped around the torus



As C winds, sweeping out the torus and returning to original loop, the phase of $W(C)$ can change by $e^{2\pi i k/N}$ — then $k \pmod N$ is the $\mathbb{Z}_N$ magnetic flux on the torus.



It is convenient to describe this in a different way...



On the periodically identified box, gauge invariant quantities should be smooth, but the connection A_μ need not be single valued: it can be periodic up to a gauge transformation

$$A(a_1, x_2) = A^{\Omega_1(x_2)}(0, x_2)$$

$$A(x_1, a_2) = A^{\Omega_2(x_1)}(x_1, 0)$$

Then we can express $A(a_1, a_2)$ in terms of $A(0,0)$ in two different ways

$$A(a_1, a_2) = A^{\Omega_2(a_1)}(a_1, 0) = A^{\Omega_1(0)\Omega_2(a_1)}(0,0)$$

or

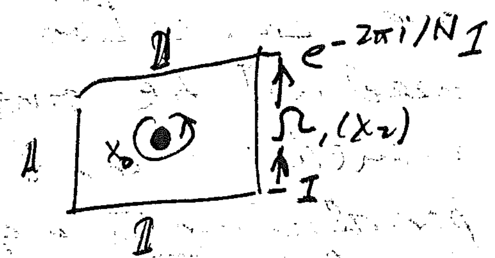
$$A(a_1, a_2) = A^{\Omega_1(a_2)}(0, a_2) = A^{\Omega_2(0)\Omega_1(a_2)}(0,0)$$

Consistency requires that $\Omega_1(0)\Omega_2(a_1)$ and $\Omega_2(0)\Omega_1(a_2)$ are equivalent as $SU(N)/Z_N$ transformations — i.e. differ by an element of the center of $SU(N)$ (cocycle condition)

$$\Omega_1(0)\Omega_2(a_1) = \exp[-2\pi i n_{12}/N]\,\Omega_2(0)\,\Omega_1(a_2)$$

or $$\Omega_1(0)\Omega_2(a_1)\Omega_1(a_2)^{-1}\Omega_2(0) = \exp(-2\pi i n_{12}/N)$$

This element of the center is a topological invariant of the bundle, and in fact it is just the magnetic flux



E.g. suppose an 't Hooft vortex creation operator acts on the torus.

In a particular gauge, the 't Hooft gauge transf. $\Omega(x_0)$ has no effect on $\Omega_2(x_1)$, ~~but~~ or on $\Omega_1(0)$, but it induces

$$\Omega_1(a_2) \to e^{-2\pi i/N}\,\Omega_1(0)$$

and so reduces $n_{12} \to n_{12} - 1$ — i.e. annihilates a unit of flux.

On the three-torus, there is a flux n_{ij} associated with each of the 3 planes (the homologically nontrivial surfaces). We take

$$m_i = \tfrac{1}{2}\epsilon_{ijk}\, n_{jk}$$

to be the magnetic flux in the $\hat{e}_i$ direction.

Electric Flux

What is electric flux? As discussed previously, on a torus there can be "large" time-independent gauge transformations that cannot be smoothly deformed to trivial, and so are not required by the Gauss law constraint to act trivially on physical states. These are the gauge transformations that have a Z_N winding number — i.e. are periodic up to an element of the center of the gauge group

On the 3-torus, the gauge transformation Ω has three winding numbers

$$\Omega(a_1, x_2, x_3) = e^{-2\pi i k_1/N}\,\Omega(0, x_2, x_3)$$
$$\Omega(x_1, a_2, x_3) = e^{-2\pi i k_2/N}\,\Omega(x_1, 0, x_3)$$
$$\Omega(x_1, x_2, a_3) = e^{-2\pi i k_3/N}\,\Omega(x_1, x_2, 0)$$

How a gauge transformation acts on physical states is determined only by the values of $k_{1,2,3}$ mod N — as two transformations with the same values differ by a small transformation that acts trivially on $\mathcal{H}_{physical}$.

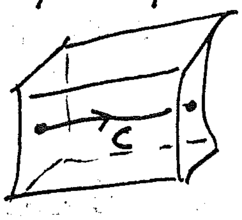
Thus we identify a $(Z_N)^3$ global symmetry for $G = SU(N)/Z_N$ gauge fields in the box, and states can be decomposed as irreducible representations of the symmetry, in which $\Omega(\vec{k})$ is represented by

$$\Omega(\vec{k})\,|\vec{e}\rangle = \exp\left(\frac{2\pi i}{N}\vec{e}\cdot\vec{k}\right)|\vec{e}\rangle$$

where $e_{1,2,3}$ are integers defined mod N

These integers are the <u>electric</u> <u>fluxes</u> in the 1,2,3 directions.

We can see that $\vec{e}$ is the electric flux by noting that large gauge transformations do not commute with the fundamental irrep Wilson loop $W(C)$:



$$W(C)\,\Omega(\vec{k}) = e^{-2\pi i k_3}\,\Omega(\vec{k})\,W(C)$$

if C winds around torus in the 3 direction (g.t. changes

$$P\exp\left(i\int_0^{a_3} A\right) \to \Omega(a_3)\, P\exp\left(i\int_0^{a_3}\right)\Omega(0)$$

So $W(C)$ boosts the eigenvalue of $\Omega(\vec{k})$ by $e_3 \to e_3 + 1$.

That is right, since we know that $W(C)$ creates a line of Z_N electric flux along C.

We can construct projection operators that project out states with definite transformation properties, and hence definite flux

$$P(\vec{e}) = \frac{1}{N^3}\sum_{\vec{k}} e^{-2\pi i \vec{k}\cdot\vec{e}/N}\,\Omega(\vec{k})$$

– Note that:

$$\Omega(\vec{k}')\,P(\vec{e}) = \frac{1}{N^3}\sum_{\vec{k}} e^{-2\pi i \vec{k}\cdot\vec{e}/N}\,\Omega(\vec{k}+\vec{k}')$$

$$= e^{2\pi i \vec{k}'\cdot\vec{e}/N}\,P(\vec{e})$$

by shifting the sum. This shows that

$\underline{P}(\vec{e})$ acting on any state gives a state with electric flux $\vec{e}$. It is a projector because

$$\underline{P}(\vec{e}\,')\,\underline{P}(\vec{e}) = \frac{1}{N^3}\sum_{\vec{k}'} e^{-2\pi i \vec{k}'\cdot\vec{e}\,'/N}\, e^{2\pi i \vec{k}'\cdot\vec{e}/N}\, \underline{P}(\vec{e})$$

$$= \delta_{\vec{e},\vec{e}\,'}\, \underline{P}(\vec{e})$$

Energy of a Flux Configuration

We will evaluate the ground state energy in a sector of definite flux by inserting into the path integral a projector onto a sector of definite flux, and studying the behavior as the box gets large.

Actually, it is convenient to introduce periodic boundary conditions in the Euclidean time direction, and so evaluate the partition function for gauge fields in a box at finite (but low) temperature.

At inverse temperature β,

$$Z(\beta) = \mathrm{tr}\, e^{-\beta H} = \sum_n \langle n | e^{-\beta H} | n\rangle$$

where $\{|n\rangle\}$ is a complete set of field eigenstates

$$= \int_{BC} e^{-S_E},$$ where the BC imposes periodicity in time direction with period β.

In a gauge theory, we need to restrict to the gauge-invariant states, which we can do by inserting a projector, or equivalently integrating over all small gauge transformations

$$Z(\beta) = \int dA_i(\vec{x}) \underbrace{d\Omega(\vec{x})}_{\substack{\text{integrated} \\ \text{over small} \\ \text{gauge transformations}}} \langle A^\Omega | e^{-\beta H} | A \rangle$$

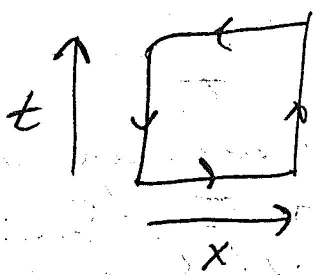
$$= \underbrace{\int_\beta (dA_i)}_{\substack{\text{integrated over histories periodic} \\ \text{in Euclidean time up to a small} \\ \text{gauge transformation.}}} e^{-S_E[A]}$$

This $Z(\beta)$ includes contributions from all flux sectors. We restrict to a particular sector by projecting onto that sector

$$Z[\vec{e}, \vec{m}; \beta] = \int d\Omega(\vec{x})\, dA_i(\vec{x}) \langle A^\Omega | e^{-\beta H} P(\vec{e}, \vec{m}) | A \rangle$$

We project out $\vec{m}$ by restricting $A_i(\vec{x})$ to obey the required boundary condition, and project out $\vec{e}$ with

$$P(e) = \frac{1}{N^3} \sum_{\vec{k}} e^{-2\pi i (\vec{e}\cdot\vec{k})/N}\, \Omega(\vec{k})$$



Note that $\Omega(\vec{K})$ acts on the initial configuration but not the final one — it has the effect of introducing a unit of Z_N magnetic flux in the x-t plane — Thus $\vec{K}$ can be interpreted as

$$n_{0i} = K_i \quad \text{— the flux in the } 0i \text{ plane}$$

And so we have

$$Z[\vec{e}, \vec{m}; \beta] = \frac{1}{N^3} \sum_{\vec{K}} e^{-2\pi i/N(\vec{K}\cdot\vec{e})} \int (dA)_{\vec{K},\vec{m}}\, e^{-S_E[A]}$$

$$= \frac{1}{N^3} \sum_{\vec{K}} e^{-(2\pi i/N)\vec{K}\cdot\vec{e}}\, W[\vec{K}, \vec{m}; \beta]$$

path integral over sector with

$$n_{0i} = K_i$$

$$m_k = \tfrac{1}{2}\varepsilon_{kij} n_{ij}$$

We will eventually take the limit

$$\lim_{\beta, \vec{a} \to \infty} Z[\vec{e}, \vec{m}; \beta] = \lim_{\beta, \vec{a} \to \infty} e^{-\beta F[\vec{e}, \vec{m}; \beta]}$$

$$\sim \exp(-\beta E_{vac}[\vec{e}, \vec{m}; \vec{a}])$$

— dominated by vacuum in the sector, <u>if</u> there is a mass gap.

Now we can use an electric-magnetic duality relation to constrain $Z[\vec{e}, \vec{m}]$. We may as well take all sides of the box to be equal: $a_1 = a_2 = a_3 = \beta$. Then $W[\vec{k}, \vec{m}]$ is invariant under Euclidean rotations of the box — if we transform $\vec{k}, \vec{m}$ suitably

E.g. consider the 4D rotation

$$\left[\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

under which

$$\begin{cases} x_1 \to -x_2 \\ x_2 \to x_1 \end{cases} \quad \begin{cases} x_3 \to x_0 \\ x_0 \to -x_3 \end{cases}$$

$$\begin{aligned} n_{01} &\to n_{32} = -n_{23} \\ n_{02} &\to -n_{31} \\ n_{03} &\to -n_{30} = n_{03} \\ \hline n_{12} &\to -n_{21} = n_{12} \\ n_{23} &\to n_{10} \\ n_{31} &\to -n_{02} = n_{20} \end{aligned}$$

Thus the rotation leaves k_3 and m_3 alone, but

$k_1 \leftrightarrow m_1$
$k_2 \leftrightarrow m_2$

Hence,

$$W[\vec{k}, \vec{m}] = W[m_1, m_2, k_3; k_1, k_2, m_3]$$

What does this imply for $Z[\vec{e}, \vec{m}]$

$$Z[\vec{e}, \vec{m}] = \frac{1}{N^3} \sum_{\vec{k}} e^{-(2\pi i/N)\vec{k}\cdot\vec{e}}\, W[m_1, m_2 k_3; k_1, k_2 m_3]$$

use the inverse transform

$$W[\vec{k}, \vec{m}] = \sum_{\vec{e}} e^{(2\pi i/N)\vec{k}\cdot\vec{e}}\, Z[\vec{e}, \vec{m}]$$

$$Z[\vec{e},\vec{m}] = \frac{1}{N^3}\sum_{\vec{k},\vec{l}} e^{-(2\pi i/N)\vec{k}\cdot\vec{e}}$$

$$\cdot \exp\left[\left(\frac{2\pi i}{N}\right)(m_1 l_1 + m_2 l_2 + k_3 l_3)\right]$$

$$\cdot Z[\vec{l}, k_1, k_2, m_3]$$

The k_3 sum imposes $l_3 = e_3 \Rightarrow$

$$\boxed{Z[\vec{e},\vec{m}] = \frac{1}{N^2}\sum_{\substack{k_1,k_2\\ l_1,l_2}} \exp\left[-\frac{2\pi i}{N}(k_1 e_1 + k_2 e_2 - l_1 m_1 - l_2 m_2)\right] \cdot Z[l_1, l_2, e_3; k_1, k_2, m_3]}$$

— the Duality relation

What are the consequences? As the box gets large

$$Z[\vec{e},\vec{m}] = e^{-\beta F[\vec{e},\vec{m}]} \longrightarrow \begin{cases} 1 & \text{"light flux"} \\ 0 & \text{"heavy flux"} \end{cases}$$

— i.e. $F \to 0$ and $Z = 1$ if there are no stable flux tubes in that sector, and $F \to \infty$ and $Z = 0$ if there are stable flux tubes.

So $Z[\vec{e},\vec{m}]$ are N^6 numbers taking values 0, 1.

What do we know about these numbers?

- We assume $(\vec{0}, \vec{0})$ is a light flux (the vacuum)

- If $(\vec{e}, \vec{m})$ is light, then $(0, \vec{m})$ is light.

This is because
$$Z[\vec{e}, \vec{m}] = \frac{1}{N^3} \sum_{\vec{k}} e^{(2\pi i/N)\vec{k}\cdot\vec{e}}\, W[\vec{k}, \vec{m}]$$

~~If Z ... a ...~~
Each $W[\vec{k}, \vec{m}]$ is nonnegative – if the vacuum angle θ is zero – since then the Euclidean path integral is positive definite. If $Z[\vec{e}, \vec{m}] = 1$, then since positivity of W implies
$$Z[0, \vec{m}] = \frac{1}{N^3} \sum_{\vec{k}} W[\vec{k}, \vec{m}] \geq Z[\vec{e}, \vec{m}],$$
we must have $Z[0, \vec{m}] = 1$, too.

- Let's also assume spatial rotational invariance:
$$Z[\vec{e}, \vec{m}] = Z[R\vec{e}, R\vec{m}]$$
if R is a rotation that preserves the 3D box. Thus
$$(\vec{e}, \vec{m}) \text{ is light} \iff (R\vec{e}, R\vec{m}) \text{ is light}$$

- From duality, we have the inequality

$$Z[\vec{e},\vec{m}] = \frac{1}{N^2}\sum_{\tilde{k}\tilde{l}} \exp\left(-\frac{2\pi i}{N} k_1 e_1 + k_2 e_2 - l_1 m_1 - l_2 m_2\right) Z[l_1 l_2 e_3; k_1 k_2, m_3]$$

$$\leq \frac{1}{N^2}\sum Z[l_1,l_2,e_3; k_1,k_2,m_3] = Z[0,0,e_3; 0 0 m_3]$$

Hence $(\vec{e},\vec{m})$ is light $\Rightarrow$ $(0 0 e_3; 0 0 m_3)$ is light

- Now supposing $(\vec{e},\vec{m})$ is light consider duality relation for $Z[0 0 e_3; 0 0 m_3]$

$$1 = Z[0 0 e_3; 0 0 m_3] = \frac{1}{N^2}\sum_{\tilde{k}\tilde{l}} Z[l_1 l_2 e_3; k_1 k_2, m_3]$$

Since the sum is 1, there must be N^2 terms in the sum with $Z = 1$

If $(\vec{e},\vec{m})$ is light, then there are N^2 light fluxes with the same e_3 and m_3

- Now look back at the duality relation for $Z[\vec{e},\vec{m}]$ again. There are exactly N^2 terms that are nonzero, hence each term must be 1, and we conclude that all phases are trivial

Hence: If $(e_1 e_2 e_3; m_1 m_2 m_3)$ and $(e_1' e_2' e_3; m_1' m_2' m_3)$ are both light, then

$$e_1 m_1' + e_2 m_2' - e_1' m_1 - e_2' m_2 \equiv 0 \pmod N$$

- Now suppose $(\vec{e}, \vec{m})$ is light, then from above, both

$$(0\,0\,e_3,\ 0\,0\,m_3)$$

and $(0\,0\,0,\ 0\,0\,m_3)$ are light

then from rotational invariance, each

$(e_i, 0, 0;\ m_i, 0, 0)$ where $j, i = 1,2,3$ i.e. e_i m_i are any component of the light flux.

$(0, 0, 0;\ m_j, 0\,0)$

therefore $\boxed{m_i e_j \equiv 0 \pmod N), \quad i, j = 1,2,3}$

Similarly
If $(\vec{e}, \vec{m})$ and $(\vec{e}\,', \vec{m}\,')$ are light, then so are

$(e_i\ 0\,0;\ m_i\ 0\,0)$
$(0, 0, 0;\ m_j'\ 0\,0)$ $\Rightarrow$ $\boxed{e_i m_j' \equiv 0 \pmod N)}$

and so are

$(e_i'\ 0\,0;\ m_i'\ 0\,0)$
$(0\ 0\,0;\ m_j\ 0\,0)$ $\Rightarrow$ $\boxed{e_i' m_j \equiv 0 \pmod N)}$

Now -- suppose N is prime (e.g. here)
Then, if $\vec{m}$ has a nonzero component for any light flux, $\vec{e} = 0$, and if $\vec{e}$ has a nonzero component, $\vec{m} = 0$.

Now, N prime, there are only 2 rotationally invariant solutions

(i) $(0, \vec{m})$ is light — This is electric confinement

(ii) $(\vec{e}, 0)$ is light — This is magnetic confinement – i.e. Higgs phase

If N not prime, there can be other solutions (HW exercise).

Naturally, we cannot rule out the Higgs phase, which really is possible with adjoint Higgs fields. And we needed the assumption of a mass gap ($Z = 0$ or 1) to rule out a Coulomb phase. E.g. $SU(2)/\mathbb{Z}_2 \to U(1)$ has neither electric or magnetic confinement

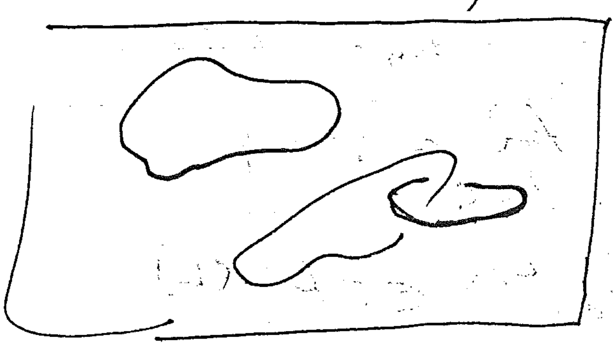
Deconfinement at Finite Temperature

Since we now know how to formulate

$$Z[\beta, \vec{a}] = \exp\left[-\beta F(\beta, \vec{a})\right]$$

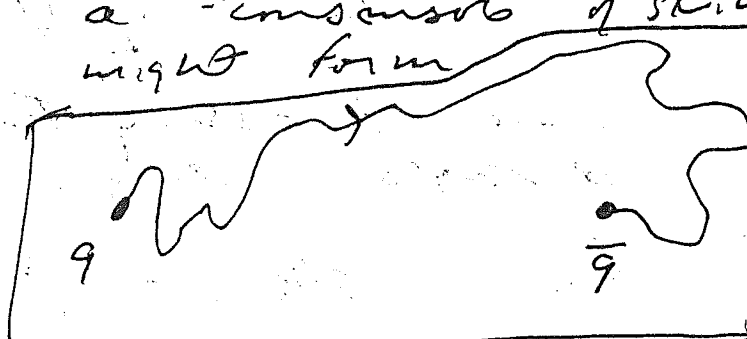
for a gauge theory, we can study the $\vec{a} \to \infty$ limit with β fixed, the bulk thermodynamics of the $SU(N)/Z_N$ gauge theory.

Suppose the theory is in the confining phase at $\beta = \infty$. What do we expect to happen as we heat the system up, i.e. increase β^{-1}?



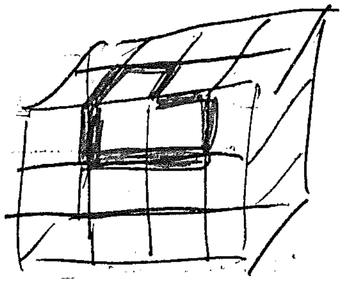
Closed loops will arise due to thermal fluctuation

At high enough temperature a "condensate" of strings might form



Then adding quark sources might not appreciably increase the free energy — adding another flux tube is not costly, if many long flux tubes are already present.

A crude model:
string bits occupy a
cubic lattice with
lattice spacing a



($a \sim$ flux tube thickness)

The number of paths on the lattice of length L, in d dimensions

$$\sim (2d-1)^{L/a}$$

Path can turn in $2d-1$ directions

Non self intersection corrects this to

$$(\mu_d)^{L/a}$$

$\mu_2 \sim 2.638$
$\mu_3 \sim 4.684$
$\mu_4 \sim 6.77$

The partition function for a gas of string loops behaves like

$$Z \sim \sum_L (\mu_d)^{L/a} e^{-\beta(KL)}$$

where K = string tension. The sum diverges (long strings become unsuppressed)

for $\quad \beta_c(KL) = \frac{L}{a} \ln \mu_d$

or $\quad \beta_c^{-1} = \dfrac{aK}{\ln \mu_d}$

In "reality" $a \sim \Lambda^{-1}$, $K \sim \Lambda^2 \Rightarrow \beta_c^{-1} \sim \Lambda$

What is a proper criterion for confinement? Finite β breaks the symmetry between spacelike and timelike loops, and there is no reason to expect them to behave in the same way...

Consider limit of high-temperature in $A_0 = 0$ gauge formulation

$$Z[\beta] = \int d\Omega(\vec{x})\, dA_i(\vec{x},t)\, \exp\left[-\frac{1}{2g^2}\int_0^\beta dt \int d^3x \left[(\dot{A}_i^a)^2 + (B_i^a)^2\right]\right],$$

$$\text{B.C.: } A_i(\vec{x},\beta) = A_i^\Omega(\vec{x},0)$$

In limit $\beta \to 0$, it is costly for fields to vary with time — costs kinetic energy

$$\sim (A_f - A_i)^2/\beta$$

As $\beta \to 0$ (high temperature), path integral is dominated by time-independent configurations and the integral over $\Omega(\vec{x})$ becomes peaked at trivial $SU(N)/Z_N$ gauge transformations — $\Omega(\vec{x})$ is nearly constant and close to an element of Z_N — the center of $SU(N)$.

Expectation values in the thermal ensemble approach those of 3dim Euclidean Yang-Mills theory, with 3-dimensional coupling

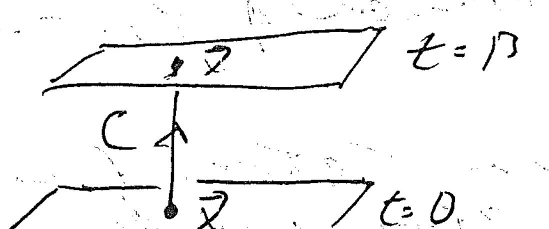
$$g_{3D}^2 = g_{4D}^2\, T \qquad (T = \beta^{-1} = \text{temp.})$$

We expect that pure Yang-Mills in 2+1 dim is confining, so spacelike loops Wilson loops in high-temp 3+1 dim YM expected to exhibit area law. Flux tubes are locally stable — but does this imply confinement?

We should consider instead the response to introducing colored sources in the thermal bath. It is convenient to express the partition function differently — in a gauge with $A_0 \neq 0$ and A periodic (rather than periodic up to a gauge transformation, in Euclidean time with period β):

$$Z[\beta] = \int (dA_\mu)_{\text{periodic}}\, e^{-(S_E)}$$

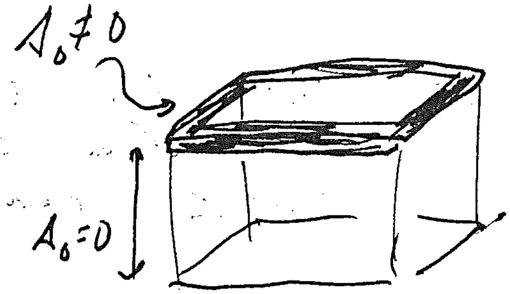
(where gauge fixing is understood, to remove infinite volume of gauge group). This becomes earlier expression if we fix "$A_0 = 0$ gauge."



Actually we can't quite fix $A_0 = 0$ everywhere, since

$$W(C) = \operatorname{tr} P \exp\left(i \int_{(\vec{x},0)}^{(\vec{x},t)} A_0\, dt\right)$$

is gauge-invariant — Wilson loop that winds around spacetime in Euclidean time direction



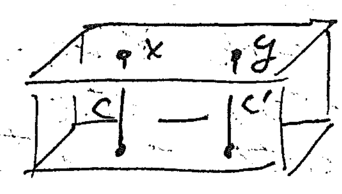
The best we can do is squeeze region where $A_0 \neq 0$ into a thin slab. Alternatively, we can set $A_0 = 0$ with a gauge transformation that is <u>not</u> periodic

$$\underline{P}\exp\left(i\int_0^t A_0\,dt\right) \Rightarrow \Omega(\vec{x},t)\left[\underline{P}\exp\left(i\int A\right)\right]\Omega^{-1}(\vec{x},0)$$

$$= I$$

$$\Rightarrow \Omega^{-1}(\vec{x},t)\,\Omega(\vec{x},0) = \underline{P}\exp\left(i\int_{(\vec{x},0)}^{(\vec{x},t)} A\right)$$

Then in this $A_0 = 0$ gauge, $A_i(\vec{x},t)$ is <u>not</u> periodic, but instead periodic up to an arbitrary gauge transformation — and the A_0 integral becomes $\int d\Omega(\vec{x})$



Now response to introducing sources at $\vec{x}$, $\vec{y}$ is probed by

$$\langle W(C)\; W(C')\rangle_\beta$$

where C, C' wind in time direction. Even more simply

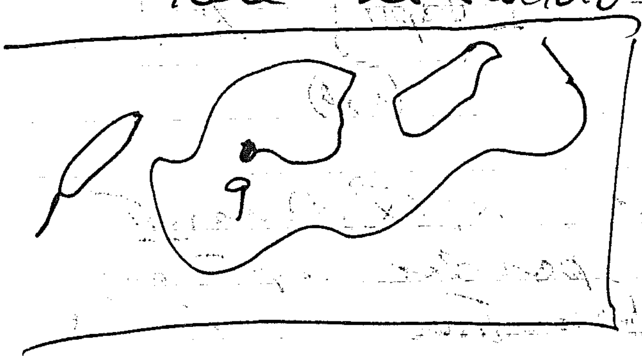
$$\langle W(C)\rangle = \exp\left(-\beta F_{quark}\right)$$

the timelike Wilson line determines the free energy cost of introducing a single classical quark source.

If quarks are confined, then $F_{quark}(\beta) \to \infty$ in limit of infinite volume, and

$$\langle W(\vec{x}) \rangle_\beta \xrightarrow[Vol\to\infty]{} 0$$

Hence $\langle W(\vec{x}) \rangle_\beta \neq 0$ signifies that there is "electric disorder" so we can introduce a quark w/o much cost



(Adding another long flux tube doesn't matter, because there are a lot of long flux tubes present already.)

As in (2+1) dimensional Yang-Mills at zero temperatures, the confinement/deconfinement transition can be interpreted as a change in the realization of a Z_N symmetry.

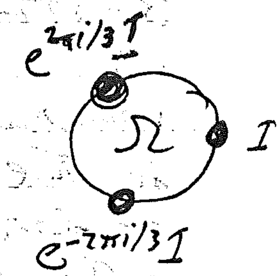
Now the Z_N consists of gauge transformations in Euclidean spacetime that are large — i.e. twist by an element of the center in the

Euclidean time direction. These gauge transformations preserve the action and path integral measure, but transform

$$W(\vec{x}) \longrightarrow e^{2\pi i K/N}\, W(\vec{x})$$

In the confining phase, $\langle W(\vec{x})\rangle = 0$, and the symmetry is manifest; in the deconfined phase $\langle W(\vec{x})\rangle \neq 0$, and the Z_N is spontaneously broken.

Furthermore, we have already argued that the integral over $\Omega(\vec{x})$ (the remnant of $W(\vec{x})$ in the $A_0 = 0$ gauge) is sharply peaked around



$$\Omega \in Z_N = \{ e^{2\pi i K/N} I,\ K = 0, \ldots, N-1 \}$$

in the limit of temperature $\Rightarrow \infty$.

This is the symptom of spontaneous breaking of the Z_N symmetry. Fluctuations in which $W(\vec{x})$ jumps from near one element of the center to near another element become suppressed.

Z_N is spontaneously broken ($\langle W(\vec{x})\rangle \neq 0$), and deconfinement occurs. However spacelike loops still decay like $e^{-\text{Area}}$, and there is no magnetic confinement either.

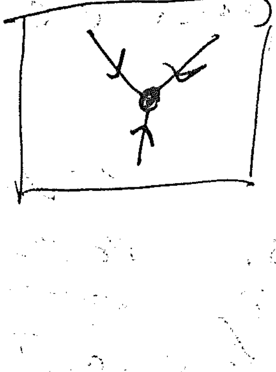
Order of the transition

Thus the deconfinement transition is in the same universality class as that of a Z_2 Ising spin system in 3 dimensions (for $G = SU(2)/Z_2$).

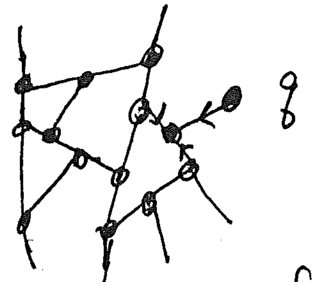
The phase transition should be second order, with Ising exponents. This is confirmed by Lattice Monte Carlo studies.

For $G = SU(3)/Z_3$, it is in the universality class of a Z_3 magnet — i.e. 3-state Potts model. There is no known fixed point of the renormalization group in this case, and so the transition is expected to be first order. This is also seen in lattice calculations.

There is a heuristic interpretation of the difference between $N=2$ and $N=3$. For $N=3$, flux tubes are directed, and can meet at vertices

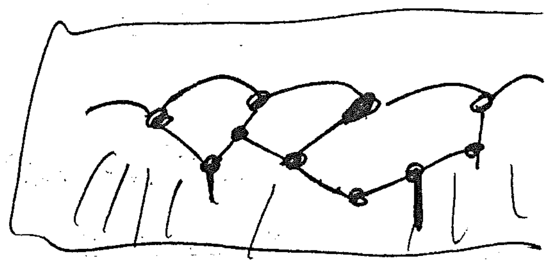


At high temp, instead of a gas of loops, the flux tubes form a percolating network that fills space



[Require even number of links on every closed loop.]

Then if we add one quark, it is pretty easy for the extra flux tube to "hook on" to the existing network, and the network carries the new unit of Z_N flux off to ∞, without much cost in free energy.

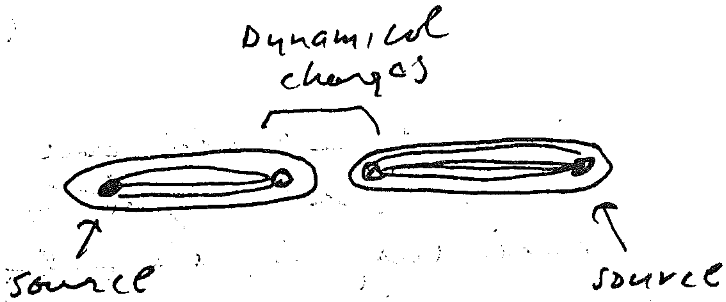


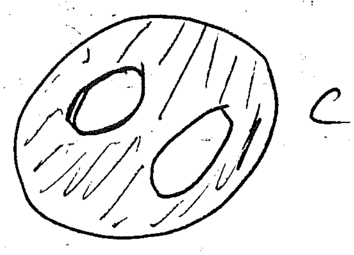
The transition is first order because phases can coexist, separated by a boundary with surface tension. (At the boundary "loose ends" are tied up, at a cost in free energy proportional to area of the interface.)

Dynamical Quarks

So far we have discussed confinement and Higgs phases in $G = SU(N)/Z_N$. But what if we introduce matter in the fundamental representation (or some other rep that does not represent the center trivially?)

With dynamical Z_N electric charges, electric flux tubes are unstable — the tube can break via nucleation of a pair of charges:





Holes appear in the string world sheet, and expand --

$$\langle W(C)\rangle \propto e^{-\mu(\text{Perimeter})}$$

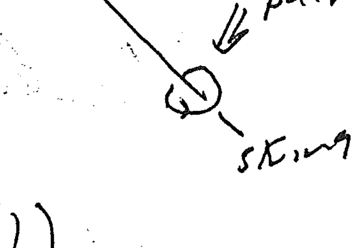
irrespective of whether there is "confinement."

What about the 't Hooft loop? It is the world line of a classical Z_N monopole, or the boundary of a Z_N dirac string.

But the fundamental irrep matter can see the string — so the string is a physical object rather than a gauge artifact

$B(C)$ becomes $B(C,\Sigma)$ — it depends on how the world sheet of the Dirac string is chosen — and the string has action proportional to its area

(virtual Z_N charges nucleate and wind around the string before reannihilating



$$\langle B(C,\Sigma)\rangle \propto \exp(-(\cdot)\,\text{Area}(\Sigma))$$

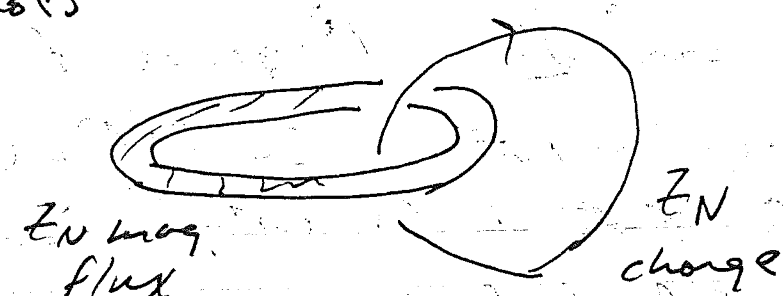
— irrespective of whether there is confinement

Nevertheless, it is possible to distinguish different realizations of the Z_N symmetry.

Recall the case of adjoint Higgs fields that completely Higgs

$$SU(N) \to Z_N$$

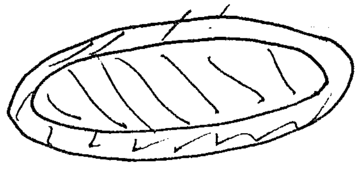
— so there are stable Z_N magnetic flux tubes



Now suppose there are also Higgs fields that carry Z_N charge.

These particles have an infinite range Aharonov - Bohm interaction with the flux tubes. Hence, in principle a Z_N charge can be detected by a string that is infinitely far away, and as we have argued previously the theory has Z_N charge superselection sectors.

What happens if the fundamental irrep Higgs field condenses? Then Z_N is spontaneously broken, and the flux tube is unstable, it becomes the boundary of a domain wall.



Hence there is a phase boundary, separating phase with a Z_N superselection rule from phase without one.

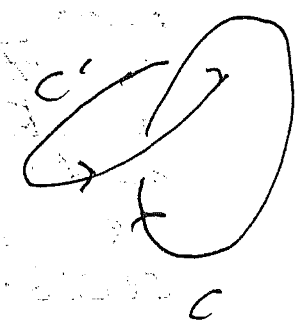
What is an order parameter for this transition? We can define an operator $F(\Sigma)$ ("flux operator") that inserts into the path integral a classical string worldsheet on Σ (gauge field is constrained so that $P\exp(i\oint_C A) = e^{2\pi i/N} I$ on path C that links with Σ.)

As already noted, field fluctuations see the string and

$$\langle F(\Sigma)\rangle \sim \exp(-\text{Area})$$

But consider

$$\langle F(\Sigma) W(C)\rangle$$



where C links with Σ. If there is a charge superselection rule, this correlator has an Aharonov-Bohm phase that is sensitive to the linking number of Σ, C.

We consider...

$$A(\Sigma, C) \equiv \frac{F(\Sigma)\, W(C)}{\langle F(\Sigma)\rangle \langle W(C)\rangle}$$

— the ABOP — or "Aharonov – Bohm order parameter" then

$$\lim \langle A(\Sigma, C)\rangle = \exp\left(\frac{2\pi i}{N} K(\Sigma, C)\right)$$

in the "free-charge" phase with Z_N superselection sectors

$K(\Sigma, C)$ is the linking number of Σ, C, and the limit is taken so that Σ, C, though linked, are arbitrarily far apart.

In the phase with no free charges

$$\lim \langle A(\Sigma, C)\rangle = 1$$

It seems like there are two distinguishabl phenomena that could destroy the Z_N superselection rule. A Higgs condensate with Z_N charge screens the Z_N charge, and destabilizes the magnetic flux tubes. Or confinement causes Z_N charges to appear only in mesons where they are tied to opposite charges (or baryons with trivial Z_N charge). How do we distinguish these two?

We don't.

There appears to be no well defined boundary that separates Higgs and Confinement phases, when there is matter transforming under Z_N

Example: $G = SU(2)$ with a scalar doublet $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$

and a nonchiral fermion doublet $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$
Let's compare the spectrum of the theory in the case of Higgs behavior and confinement behavior, to see if it is plausible for the spectrum to evolve continuously from one regime to the other.

For $v^2 > 0$, we usually describe the spectrum of this theory in the following way. We use the gauge freedom to rotate ϕ so that

$$\phi_1 = 0$$
$$\phi_2 = v + \tilde{\phi}.$$

This is the unitary gauge, in which the spectrum contains no unphysical gauge artifacts.

In this gauge, the spectrum consists of

scalar $\tilde{\phi}$

fermions ψ_1, ψ_2

massive gauge bosons W_1, W_2, W_3

(plus possible composite states bound by the (weak) gauge interaction)

For $v^2 < 0$, we expect that the theory is confining – physical states couple only to gauge-invariant (color singlet) operators. We classify the spectrum by these operators, which can create physical states when acting on the vacuum.

scalar $\phi_i^+ \phi_i$

fermions $\phi_i^+ \psi_i$
$\epsilon_{ij}\, \phi_i \psi_j$

vector bosons $\epsilon_{ij}\, \phi_i D_\mu \phi_j$
$\epsilon_{ij}\, \phi_i^+ D_\mu \phi_j^+$
$\phi_i^+ D_\mu \phi_i$

But these same gauge-invariant operators couple to our states in the Higgs phase. If we impose the unitary gauge condition

$$\phi_i^+ \phi_i \longrightarrow v\tilde{\phi} + \cdots$$
$$\phi_i^+ \psi_i \longrightarrow v\,\psi_2 + \cdots$$
$$\epsilon_{ij}\, \phi_i \psi_j \longrightarrow v\,\psi_1 + \cdots$$

$$\left.\begin{array}{l}\epsilon_{ij}\,\phi_i D_\mu \phi_j \\ \epsilon_{ij}\,\phi_i^+ D_\mu \phi_j^+\end{array}\right\} \rightarrow \quad v^2 W_\mu^{1,2} + \cdots$$

$$\phi_i^+ D_\mu \phi_i \quad \longrightarrow \quad v^2 W_\mu^3 + \cdots$$

we find no reason to believe that the spectrum behaves discontinuously when v^2 changes sign.

One might imagine trying to distinguish the Higgs and confinement phases by a clever trick: couple the scalar and fermion to a $U(1)$ gauge field, through the current

$$J_\mu = \frac{e}{2}\left(\phi^+ \overleftrightarrow{D}_\mu \phi + \bar{\psi}\gamma_\mu \psi\right)$$

Now the elementary scalar and fermion have charge $e/2$, but the gauge-invariant states have charge ne. Can't we then tell whether the elementary particles are liberated by the Higgs mechanism?

No, because $\langle\phi\rangle = \binom{0}{v}$ breaks $SU(2)\times U(1)$ down to a new $U(1)$ with

$$Q' = Q + T_3$$

under which the elementary fields carry integer charge. The expectations for the charges of the physical states are the same in the Higgs picture and the confinement picture. ($Q' = Q$ for a gauge invariant operator, and v has $Q' = 0$.)

Now contrast the above discussion with the case of a model with a scalar in the triplet representation of $SU(2)$ $\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$

If there are no fermions in the defining rep, then we can define $B(C)$, and also expect $A(C)$ to be a reasonable order parameter for confinement, because color electric Z_2 charge cannot be screened.

Thus, it is possible to distinguish a Higgs phase from a confinement phase. And the distinction survives even when we introduce the fermion doublet $\binom{\psi_1}{\psi_2}$, as we see by considering the spectra

In the Higgs phase, we expect $SU(2) \to U(1)$ and we have, in unitary gauge

$$\phi_1 = \phi_2 = 0$$
$$\phi_3 = v + \tilde{\phi}$$

Scalar $\tilde{\phi}$ (neutral)
Fermions ψ_1, ψ_2 (charge $\pm\frac{1}{2}$, unconfined)
Massive $W^\pm$
Massless W^0

But in the confinement phase, we have
Scalars $\phi_i\phi_i$, $\bar{\psi}_i\psi_i$
Massive Vector $\phi_i D_\mu \phi_i$
There are No fermions

Because states with odd fermion number have nontrivial Z_2, and physical states are all trivial under Z_2. Apparently, there is a phase transition.

What is the fundamental difference between these two examples. Clearly it is the existence of a surviving Z_N (Z_2) symmetry which can be used to classify states. When we say that the gauge symmetry is broken, we mean that there exist physical states which transform nontrivially under Z_N. (In the $SU(2)$ model with a Higgs triplet, Z_2 happens to coincide with $(-1)^F$ where F is fermion number) We cannot make sense of this condition, however, if the unitary gauge order parameter transforms nontrivially under Z_N. Then all states have indefinite Z_N quantum numbers

Consider now the implications for the deconfining transition in the presence of dynamical quarks. The "pure gauge theory limit" is $m = \infty$, since for infinite mass quarks, breaking of the string can be ignored. For $m < \infty$, the Z_N global symmetry discussed on p 2.48 is explicitly broken; quarks, transforming as the defining representation of $SU(N)$, satisfy boundary conditions which are not left-invariant by the $SU(N)/Z_N$ gauge transformations considered there.

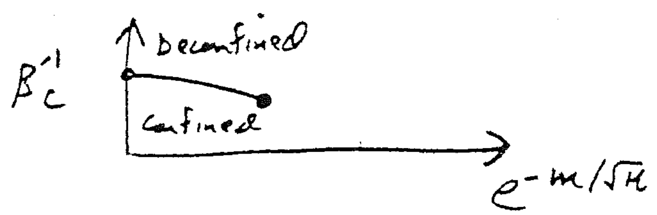
i) $N = 2$

Neglecting dynamical quarks, we argued that large closed loops of electric flux would "condense" at a temperature $\beta^{-1} \sim \sqrt{\kappa}$. In this temp. range, the thermal estimate of the "tunneling" probability applies.

No matter how large m is, breaking of the string will prevent arbitrarily large loops from forming. It seems reasonable to guess, then, that the 2nd-order phase transition will disappear for $m < \infty$. This is consistent with standard wisdom; 2nd-order phase transitions are typically unstable under small perturbations. (E.g., Ising magnet in external field)

ii) $N \geq 3$

For $m = \infty$, we argued that the deconfining transition, driven by percolation of a string network, was first order. For m large, the time scale $\sim e^{2m\beta}$ for a link in the network to break is large compared to the characteristic time scale for fluctuations of the network. So a phase transition can still be expected. The order parameter $\langle \mathrm{tr}\, U \rangle$, while nonzero in both phases, is discontinuous at some critical temp depending on m.



Since no symmetry distinguishes the two phases (flux tube screening and Debye screening), it is possible for the transition line to terminate at