incompressible state with $\nu = \frac{1}{2m+1}$
this is the state that our Chern-Simons theory describes.

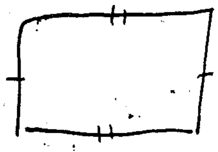
The main task of (Laughlin's) microscopic theory is to explain how, for special values of the filling factor ν, the electrons manage to find an incompressible collective state with a gap. Then the low momentum behavior is well described by the Chern-Simons field theory. It is similar to the Ginzburg-Landau theory of superconductivity (abelian Higgs model) which describes a superconductor well, but can be justified only by a microscopic (Bardeen-Cooper-Schrieffer) theory that explains the origin of the gap.

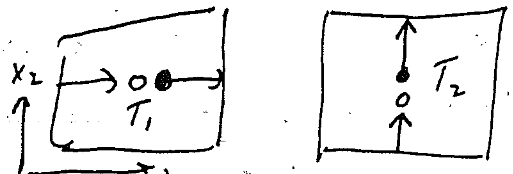
Topological Degeneracy — Lecture #7

As we will discuss in more detail later, the phases of a gauge theory cannot be distinguished by means of a local order parameter — a nonlocal criterion is needed. One such criterion is the degeneracy of the groundstate on a space of nontrivial topology.

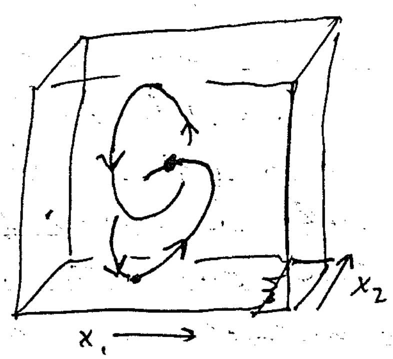
In fact, we can see that any system with anyonic excitations (whether a gauge theory or not) has such a topological degeneracy, if the anyons can be pair created.

Consider an anyonic system on a torus, a periodically identified 2-dimensional box —





Let T_1 denote a process in which we create a pair of anyons, wind one member around the pair, and then reannihilate, in the x_1 direction. T_2 is the same, in the x_2 direction.



These operations are noncommuting. The operation

$$T_2^{-1}T_1^{-1}T_2T_1$$

can be deformed to the process shown, in which the two loops (anyon worldlines) have linking number -1.

Hence: $$T_2T_1 = e^{-2i\theta}T_1T_2$$

The interpretation is that T_1 represents the Aharonov-Bohm phase accumulated by an anyon that traverses one cycle of the torus, while T_2 is the phase accumulated during traversal of the other cycle. These don't commute: carrying an anyon around C_1 alters the phase acquired by an anyon that winds around C_2.

T_1, T_2 commute with the Hamiltonian of the theory (they are symmetries) and so can be simultaneously diagonalized with it. But the T_1, T_2 algebra has no one-dimensional irreps -- so that all levels, including the ground state, are degenerate

Suppose we diagonalize the unitary operator T_1

$$T_1|\alpha\rangle = e^{i\alpha}|\alpha\rangle$$

then $T_1(T_2|\alpha\rangle) = e^{2i\theta}T_2T_1|\alpha\rangle = e^{2i\theta}e^{i\alpha}(T_2|\alpha\rangle)$,

so applying T_2 to $|\alpha\rangle$ advances the phase by

$\alpha \to \alpha + 2\theta$ ⇝ a different value modulo 2π for $\theta \neq \pi \cdot \text{integer}$

If $\theta = \frac{\pi p}{q}$ where p & q are integers with no common factor, then T_1 must have at least q distinct eigenvalues:

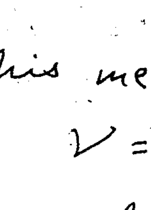
n = degeneracy

$$\Rightarrow 2\theta n = 2\pi m \Rightarrow n = \frac{\pi}{\theta}m = \pi\frac{q}{\pi p}m$$

$$\Rightarrow n = \frac{q}{\cancel{p}} \cdot \text{integer}$$

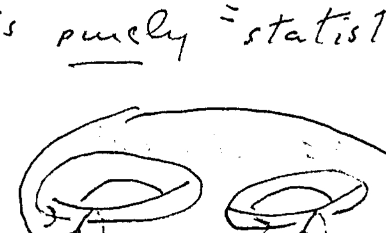
$$= q \cdot \text{integer}$$

This means that a Laughlin state with filling $\nu = p/q$, and hence $\theta = \pi p/q$, must have ground state degeneracy = multiple of q on the torus. This is strictly true only in the limit volume $\to \infty$ (and because there is a mass gap), as in that case



the linking world lines can be arbitrarily far apart (compared to correlation length), so that interaction is <u>purely</u> "statistical."

On a Riemann surface of genus g, this degeneracy behaves like $(q)^{\text{genus}}$

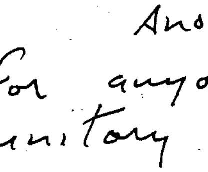


because there is a $T_{1,2}$ associated with the complementary cycles about each handle

Note that we require the anyonic phase to obey $\frac{\theta}{\pi}$ = rational = p/q for the degeneracy to be finite. So we expect θ/π = rational in "reasonable" physical systems

Another way to say what we have found: for anyons on the torus, we must obtain a unitary irrep not of the braid group of the plane, but instead the braid group of the torus, which has additional generators and defining relations. In a one-dim irrep, the constraint $T_1 T_2 = e^{i2\theta} T_2 T_1$ arises from a defining relation. - $T_{1,2}$ are A-B phases associated with transport around cycles

These considerations are general, but how do they relate to (abelian) Chern-Simons theory? Why should θ/π be rational?



In the C-S theory, our operators $T_{1,2}$ correspond to "Wilson line" operators associated with the nontrivial cycles of the torus

$$T_{1,2} = \exp\left(i \oint_{C_1, C_2} A_\mu dx^\mu\right) \equiv W(C_{1,2})$$

why should these operators be noncommuting?

Canonical Quantization of Gauge Theories

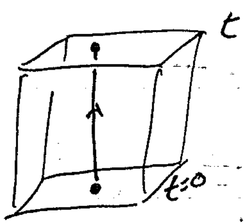
We can understand the operator algebra of the Wilson loop operators if we quantize the Chern-Simons theory via canonical commutators. To get oriented, let's recall how canonical quantization works in a conventional (abelian) gauge theory.

E.g. consider electrodynamics with

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}(\psi, D_\mu\psi)$$

$$D_\mu = \partial_\mu + ieA_\mu$$

We can make a gauge choice $A_0 = 0$.



Under a g.t. $\psi \to e^{ie\Lambda(x)}\psi$

$$A_\mu \to A_\mu + \partial_\mu\Lambda$$

$A_\mu \to 0$ where $\partial_\mu\Lambda = -A_\mu$

Draw a line in the $\hat{t}$ direction and integrate along it from $(\vec{x},0)$ to $(\vec{x},t)$

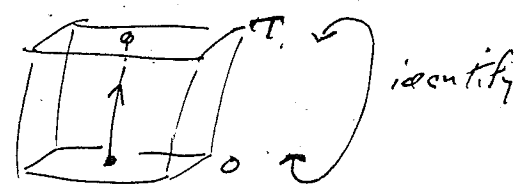
$$\Lambda(\vec{x},t) = -\int_0^t dt'\, A_0(\vec{x},t')$$

$$\psi(\vec{x},t) \to \exp\left[-ie\int_0^t dt'\, A_0\right]\psi(\vec{x},t)$$

this transforms $A_0 \to 0$

Note: $\psi^\dagger(\vec{x},t)\exp\left[ie\int_0^t dt' A_0\right]\psi(\vec{x},0)$ is invariant.

(It wouldn't be possible to transform A_0 to 0 if time were a periodic variable; then $e^{ie\oint_0^T A_0 dt}$ would be gauge invariant quantity)



When we set $A_0 = 0$, we must not forget about the A_0 equation of motion. Since there are no time derivatives of A_0 in the action, this is not a dynamical equation, but an equation of constraint that imposes a condition on the components A_i at each fixed time. The field eqn

$$0 = \frac{\delta}{\delta A_0(\vec{x},t)} S = \partial_i F^{0i} + J^0$$

where $J^0 = \frac{\delta}{\delta A_0} S_{matter}$

This is the Gauss law.

Now $\mathcal{L} = \frac{1}{2}(\dot{A}_i)^2 + \mathcal{L}_{matt}$

so that the momentum conjugate to A_i is $\Pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \dot{A}_i = E^i$ — the electric field

To quantize, we impose canonical commutators

$$[A_i(\vec{x}), E^j(\vec{y})] = i\,\delta_i^j\,\delta(\vec{x}-\vec{y})$$

and
$[A, A] = 0$
$[E, E] = 0$

In the Schrödinger representation, we may represent a state of the gauge field as a wave functional $\Psi[A_i]$

and the canonical momentum is

$$E^i(\vec{x}) = -i\,\frac{\delta}{\delta A_i(\vec{x})}$$

— a functional derivative

"Big Hilbert space": spanned by the simultaneous eigenstates of the $A_i(\vec{x})$'s — a complete set of commuting observables

Not all $\Psi[A]$ are valid physical states — we must impose the Gauss Law constraint

$$(\partial_i E^i - J^0)\,\Psi = 0$$

E.g. consider the pure gauge theory

$$\partial_i E^i(\vec{x})\,\Psi[A] = 0 \Rightarrow \int dV\,\omega(\vec{x})\,\partial_i E^i(\vec{x})\,\Psi[A] = 0$$

$$\Rightarrow 0 = \left[\int_{\text{Boundary}} d\vec{S}\cdot\vec{E}\,\omega - \int \partial_i\omega(\vec{x})\,E^i(\vec{x})\right]\Psi[A]$$

If we have $\omega(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$, so there is no surface term, then

$$0 = -\int \left(\partial_i\omega(\vec{x})\right)\left(-i\frac{\delta}{\delta A_i(\vec{x})}\right)\Psi[A]$$

suppose that ω is an infinitesimal quantity. Under a gauge transformation $A \to A + \partial\omega$

$$\delta_\omega \Psi[A] = \Psi[A+\partial\omega] - \Psi[A] = \int \partial_i\omega\,\frac{\delta}{\delta A_i}\Psi$$

so the Gauss law constraint is

$$\delta_\omega \Psi[A] = 0$$

the physical states are invariant under (time-independent) infinitesimal gauge transformations that vanish at spatial infinity... the "little" gauge transformations

And the same is true if we include matter, the role of J^0 is to implement the gauge transformation acting on the matter fields

Thus, in the quantum theory, the physical states — those obeying Gauss law, are gauge invariant states.

It is also interesting to consider $\omega(\vec{x}) \to$ nonzero constant ω_0 as $|\vec{x}| \to \infty$

then we find::

$$0 = \left[\omega_0 \int d\vec{S} \cdot \vec{E} + i\,\delta_{\omega_0}\right] \Psi_{phys}[A]$$

the way this transformation acts is determined only by ω_0 — since two transformations with the same ω_0 differ by a trans that acts trivially on physical states.

or $\left[\delta_{\omega_0} - i\omega_0 Q\right] \Psi_{phys} = 0$

where $Q = \int d\vec{S} \cdot \vec{E}$
— the electric flux at ∞

Thus physical states are not invariant under global gauge transformations, those that act nontrivially at infinity.

By building a finite transformation from infinitesimal ones, we find.

$$\omega_0: \ \Psi \to e^{iQ\omega_0}\,\Psi$$

if $\omega_0 = \omega(|x| = \infty)$ has a finite (and constant) value

Matter fields transform as $\psi(\vec{x}) \to e^{ie\omega(\vec{x})}\psi(\vec{x})$
so a transformation with
$\omega_0 = 2\pi/e \cdot$ integer is expected to act trivially

Therefore we expect $e^{2\pi i Q/e} = 1$

acting on physical states -- the compactness of the gauge group (it is $U(1)$, not $\mathbb{R}$) implies that charge is quantized.

The global gauge transformations form a group $U(1)$, and physical states transform as a representation -- if we normalize ω

$U(1) = \{e^{i\omega}, \omega \in [0, 2\pi)\}$, then ω is represented by

$$U(\omega) = e^{i\omega(Q/e)}$$

The irreducible representations are labeled by an integer $n = Q/e$. A global gauge transformation should not produce physically observable effects -- correspondingly there is a superselection rule -- all physical observables commute with $U(\omega)$ and hence preserve Q. This is a general way to describe superselection rules -- they are characterized by irreducible representations of global gauge symmetries.

[of course, the Gauss law constraint gives us an alternative way to express

$$Q = \int d\vec{S}\cdot E = \int dV\, \vec{\nabla}\cdot\vec{E} = \int dV\, J^0 .]$$

The Z_N "quantum hair" associated with a discrete gauge symmetry can also be related to a transformation property under global gauge transformations, as we will discuss later.

Okay, now let's see how canonical quantization works in Chern-Simons theory, starting with the nondynamical theory

$$\mathcal{L} = -\frac{1}{2}\mu\, \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

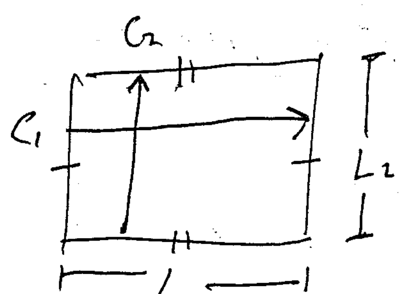
(in a homework exercise, you'll consider what happens if we also include the $-\frac{1}{4}F^2$ term).

The field equation is just $F_{\mu\nu} = 0$, and in particular, the constraint equation in the $A_0 = 0$ gauge is $F_{12} = 0$. The constraint says that the gauge connection is <u>flat</u> (has no curvature), and on the infinite plane, all flat connections are gauge equivalent to

$$A_1 = A_2 = 0$$

— any connection with vanishing curl is a gradient $A_i = \nabla_i \Lambda$ — which can be "gauged away"

But this theory is a little more interesting on $M \times \mathbb{R}$, where M is a nontrivial two dimensional manifold, like the torus $T^2 = S^1 \times S^1$.



Now the "Wilson lines" associate with nontrivial cycles of the torus allow us to identify a family of flat connections

Suppose that

$$W(C_1) = \exp\left(ie\oint_{C_1} A\right) \equiv e^{i\theta_1}$$

$$W(C_2) = \exp\left(ie\oint_{C_2} A\right) \equiv e^{i\theta_2}$$

are the observables of this theory. A flat connection A satisfying these identities is gauge equivalent to

$$A_1 = \frac{\theta_1(t)}{eL_1} \quad ; \quad A_2 = \frac{\theta_2(t)}{eL_2}$$

The action in $A_0 = 0$ gauge can be expressed as

$$S = -\frac{\mu}{2}\int d^3x \; \varepsilon^{j0i} A_j \dot{A}_i$$

$$= -\frac{\mu}{2}\int d^3x \left(A_2\dot{A}_1 - A_1\dot{A}_2\right) = \frac{\mu}{2e^2}\int dt \left(\theta_1\dot{\theta}_2 - \theta_2\dot{\theta}_1\right)$$

The equation of motion is $\frac{\partial L}{\partial \theta_1} = \dot{\theta}_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = -\dot{\theta}_2$

$\Rightarrow \dot{\theta}_{1,2} = 0$ <u>No dynamics</u>

Conjugate momenta are

$$P_1 = \frac{\partial L}{\partial \dot{\theta}_1} = -\frac{\mu}{e^2}\theta_2$$

$$P_2 = \frac{\partial L}{\partial \dot{\theta}_2} = \frac{\mu}{e^2}\theta_1$$

so the canonical commutation relations become:

$$[\theta_1, P_1] = i = -\frac{\mu}{e^2}[\theta_1, \theta_2]$$

$$[\theta_2, P_2] = i = \frac{\mu}{e^2}[\theta_2, \theta_1]$$

We conclude that

$$\boxed{[\theta_1, \theta_2] = -i\frac{e^2}{\mu} = -2i\theta}$$

where θ is the anyonic phase of a charge e object in the CS theory

Defining
$$T_1 = W(C_1) = e^{i\theta_1}$$
$$T_2 = W(C_2) = e^{i\theta_2}$$

we have $T_1 T_2 T_1^{-1} T_2^{-1} = e^{-[\theta_1,\theta_2]} = e^{2i\theta}$

(since $e^A e^B e^{-A} e^{-B} = e^{[A,B]}$, if $[A,B]$ commutes with A and B)

— this is just the result we anticipated, arising from the canonical commutation relations

Incidentally, if we Legendre transform to construct the Hamiltonian from e.g.
$$L = \theta_1 \dot{\theta}_2 \rightarrow H = p_1 \dot{\theta}_2 - L = 0$$
— the Hamiltonian vanishes, consistent with our observation that there is no dynamics. The theory is said to be a topological field theory, meaning that

- $H = 0$
- the action $\int A \wedge F$ is defined even if the manifold (in this case $T^2 \times \mathbb{R}$) is not equipped with a metric

Large Gauge Transformations

If the observables are the Wilson loops, then the variables $\theta_{1,2}$ are periodic modulo 2π, and therefore, there is a residual gauge freedom:

$$\theta_1 \to \theta_1 + 2\pi m_1$$
$$\theta_2 \to \theta_2 + 2\pi m_2$$

where m_1, m_2 are integers. These are large gauge transformations, meaning that they cannot be built from infinitesimal gauge transformations or (equivalently) they cannot be smoothly deformed to a trivial gauge transformation.

A "large" gauge transformation should not be confused with a "global" gauge transformation. A global gauge transformation on $\mathbb{R}^2$ is one that acts trivially on the "boundary" at $r \to \infty$. There are no global gauge transformations on the torus because there is no boundary. On the other hand, there are no large gauge transformations on the plane, because any

$$U(\vec{x}) = \exp(ie\omega(\vec{x}))$$

can be smoothly deformed to $U(\vec{x}) = I$.

But since the gauge group is $U(1) = S^1$, a gauge transformation on $T^2 = S^1 \times S^1$ is a map

$$S^1 \times S^1 \to S^1,$$

a map that has two integer-valued winding numbers: these are m_1 and m_2. The large gauge transformations generate the group

$$Z \times Z = \{\text{All g.t.'s}\}/\{\text{Little g.t.'s}\}.$$

and because they are not build up from infinitesimal transformations, the way that physical states transform under this $Z \times Z$ is not necessarily determined by the Gauss law constraint.

Let's construct the operators that generate $Z \times Z$

$$U_1 : \quad \theta_1 \to \theta_1 + 2\pi$$

$$U_2 : \quad \theta_2 \to \theta_2 + 2\pi$$

Representing $[\theta_1, p_1] = i$ by $\theta_1 = i \frac{\partial}{\partial p_1}$, we see that

$$\theta_1 e^{-2\pi i p_1} = 2\pi e^{-2\pi i p_1} + e^{-2\pi i p_1} \theta_1$$

Hence

$$\theta_1 |a_1\rangle = a_1 |a_1\rangle$$

$$\Rightarrow \theta_1 e^{-2\pi i p_1} |a_1\rangle = (2\pi + a_1) e^{-2\pi i p_1} |a_1\rangle$$

and so we have, from $p_1 = -\mu/e^2 \, \theta_2$
$p_2 = \mu/e^2 \, \theta_1$

$$U_1 = e^{-2\pi i p_1} = \exp\left[2\pi i \frac{\mu}{e^2} \theta_2\right]$$

$$U_2 = e^{-2\pi i p_2} = \exp\left[-2\pi i \frac{\mu}{e^2} \theta_1\right]$$

Now comes a surprise: $e^A e^B e^{-A} e^{-B} = e^{[A,B]}$ and $[\theta_1, \theta_2] = -i e^2/\mu$ imply that

$$U_1 U_2 U_1^{-1} U_2^{-1} = \exp\left[i 4\pi^2 \frac{\mu}{e^2}\right]$$

– thus the large gauge transformations U_1, U_2 do not commute, unless the "mass" μ obeys a quantization condition:

$$\mu = e^2/2\pi \cdot \text{integer}$$

Classically, the large gauge transformations commute; their failure to commute in the quantum theory is an example of an anomaly – a modification of a symmetry algebra due to a quantum effect. The same anomaly occurs if we consider the theory (on the torus) with the gauge field kinetic term, or a coupling of the gauge field to matter. In a more symmetric notation, we may define $\nu = \frac{2\pi\mu}{e^2}$

then
$$T_1 T_2 T_1^{-1} T_2^{-1} = \exp(ie^2/\mu) = e^{2\pi i/\nu}$$
$$U_1 U_2 U_1^{-1} U_2^{-1} = \exp(2\pi i \cdot 2\pi\mu/e^2) = e^{2\pi i\nu}$$

If we put the $\hbar$ back in, we would have $\nu = \frac{2\pi\mu}{e^2}\hbar$, and would see that the anomaly formally disappears in the limit $\hbar \to 0$, the classical limit

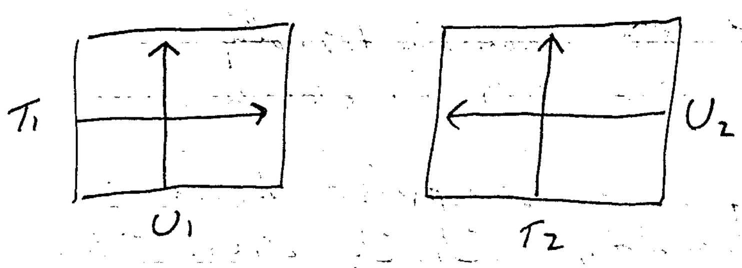
The anomaly is a surprise because the algebra generated by U_1, U_2 does not have one-dimensional representations – thus states are degenerate ($U_{1,2}$ commute with Hamiltonian) and the "gauge transformation" changes a state to another state in the same multiplet. This seems odd because gauge transformations should not have physically detectable effects

On the other hand, it becomes less of a shock if we try to reconcile the "anomaly" with our findings concerning the quantum statistics of the vortices in our phenomenological theory of the FQHE states.

Lecture #8

As the operator T can be interpreted as a process in which a pair of particles of charge e, and flux e/μ, are produced, and wind around a cycle before reannihilating, the operator U can be interpreted as a process in which a vortex carrying the flux quantum $\Phi_0 = 2\pi/e$, and charge $\mu\Phi_0 = 2\pi\mu/e$, is pair produced and winds around a cycle before reannihilating. The operators U_1 and U_2 fail to commute precisely when the vortices are anyons.

$$\frac{\theta_{\text{vortex}}}{\pi} = \nu \quad \text{and} \quad \frac{\theta_{\text{charge}}}{\pi} = \frac{1}{\nu}$$



We actually found that

$$U_1 = e^{2\pi i \frac{\mu}{e^2}\theta_2} = T_2^{\nu}$$

$$U_2 = e^{-2\pi i \frac{\mu}{e^2}\theta_1} = \left(T_1^{-1}\right)^{\nu}$$

Hence, if ν is a rational number p/q (where p, q have no common factor), this relation is telling us that q vortices are equivalent to p "electrons." Indeed, in the Laughlin states we found $\nu = 1/q$ (q odd), and that an electron could be regarded as a composite of q vortices

Therefore, if ν is rational ($\nu = p/q$), a reasonable interpretation of the "anomaly"

$$U_1 U_2 U_1^{-1} U_2^{-1} = e^{-2\pi i p/q}$$

is that the real large gauge transformations are powers of U: U_1^q and U_2^q, which commute. This corresponds to quantum tunneling around a cycle by a composite of q vortices, which is not an anyon. If the gauge transformations are $U_{1,2}^q$, then we can admit as observables

$$T_{1,2}^{1/q} = \exp\left(\frac{ie}{q}\oint_{C_{1,2}} A\right),$$

as these commute with the (new) large gauge transformations. In fact, in the case $p = 1$ (Laughlin states)

$$T_2^{1/q} = U_1$$

So the tunneling by vortices is included in the theory, but now interpreted as a Wilson loop, not a gauge transformation

We had $\nu = \frac{2\pi\mu}{e^2} = p/q$

So $$\mu = \frac{e^2}{2\pi}\, p/q = \frac{(e/q)^2}{2\pi}\, p\cdot q$$

— our "mass quantization" condition is still satisfied, but where now e is replaced by (e/q), the actual quantum of charge in the theory.
The actual physical meaning of the anomaly is that if charge is quantized in the CS theory, then

the mass must obey a corresponding quantization condition.

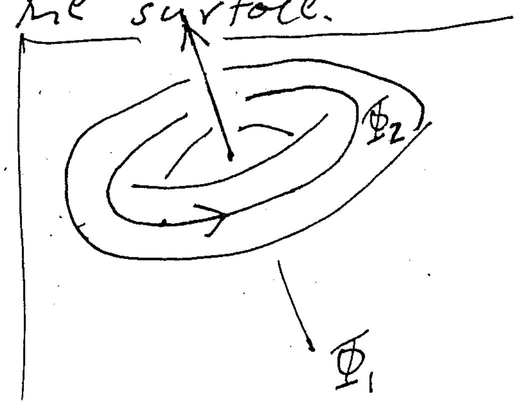
If ν is not rational, then the states of the theory are infinitely degenerate, and charge is unquantized, because flux quanta and fundamental particles have incommensurate charges. Perhaps it is fair to regard the theory as sick in that case, as there are an infinite number of "flat connections" with specified $W(C_{1,2}) = e^{i\theta_{1,2}}$,

$$\theta_{1,2} \to \theta_{1,2} + 2\pi m_{1,2}$$

are all distinct configurations, and when we sum them all, the path integral <u>diverges</u>. Anyway, the gauge group becomes $\mathbb{R}$ instead of $U(1)$.

Is the degeneracy of the ground state of the Chern-Simons theory associated with a spontaneously broken (discrete) symmetry. Not in a conventional sense. The degeneracy of the Ising model, for example, does not depend on the genus of the Riemann surface on which the spins reside (Indeed, on the sphere or plane, there is no degeneracy for the CS system.) But there is a sort of "topological symmetry" that depends on the genus of the surface.

An alternative interpretation of $\theta_{1,2}$ is that we can imagine turning on fluxes $\Phi_{1,2}$ that "link" the torus. Each flux



may start at zero and adiabatically increase to the flux quantum $\Phi_0 = 2\pi/e$. Since inserting Φ_0 is just a gauge transformation, it commutes with the Hamiltonian, and the adiabatic process preserves the ground state. But what we found is that this symmetry actually takes the ground state to a different, orthogonal, state. We need to apply the symmetry operation q times before the orbit of the state closes — in this sense there is a topological Z_q symmetry that is spontaneously broken, or $(Z_q)^{\text{genus}}$ in general.

E.g. if invariant under U_1, can't be invariant under U_2. Symmetry can't be restored...

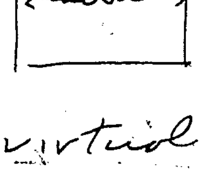
Lifting of the degeneracy

Topological degeneracy is exact in an infinite system, but is lifted slightly by finite-size effects

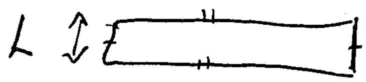
Cf. the double well potential. There are two degenerate ground states if the barrier is impenetrable, but the degeneracy is lifted by quantum tunneling. The WKB amplitude has the form $e^{-W/\hbar}$, so the effective semiclassical Hamiltonian for the low-lying states has the for

$$H = \begin{pmatrix} 0 & e^{-W} \\ e^{-W} & 0 \end{pmatrix}$$

Energy splitting $\sim 2e^{-W}$

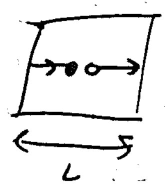


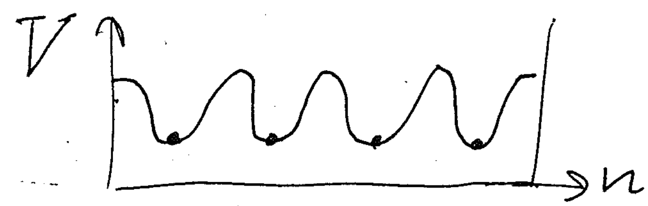
The tunneling amplitude that connects one ground state to another in the anyon system is associated with a virtual quasiparticle that travels around the cycle of the torus.



Consider a stretched torus, so that tunneling in one direction dominates...

In a relativistic theory, the amplitude for tunneling is $\exp(-mL)$, where m is the mass of anyon (or anti-anyon)





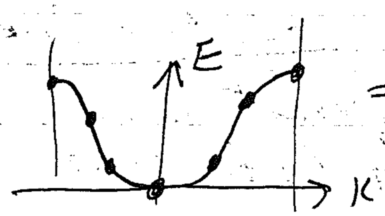
$$H = c\begin{pmatrix} 0 & e^{-S} & 0 & e^{-S} \\ e^{-S} & 0 & e^{-S} & 0 \\ 0 & e^{-S} & 0 & e^{-S} \\ e^{-S} & 0 & e^{-S} & 0 \end{pmatrix}$$

Eigenstates are

$$|k\rangle = \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} e^{-2\pi i k n/q} |n\rangle$$

$$H|n\rangle = Ce^{-S}\left(|n-1\rangle + |n+1\rangle\right)$$

$$H|k\rangle = ce^{-S}\left(e^{2\pi i k/q} + e^{-2\pi i k/q}\right)|k\rangle$$



$$\Rightarrow E_k = 2ce^{-S}\cos\left(2\pi k/q\right)$$

The splitting $\sim e^{-mL}$, becomes exponentially small as L gets large

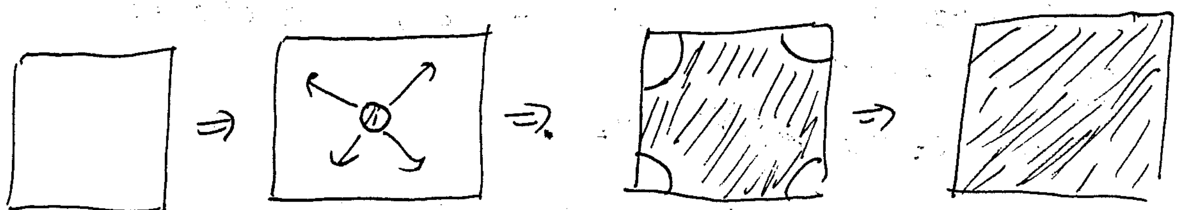
In the nonrelativistic case, there is a gap Δ, the energy required to create a pair, and a quasi-particle effective mass m^* with no simple relation to Δ. As we'll discuss later, we estimate the tunneling amplitude by finding stationary value of the Euclidean action:

$$S = T\left(\Delta + \frac{1}{2}m^*\left(\frac{L}{T}\right)^2\right) \quad \text{is}$$

stationary for $2T^2 = \dfrac{m^* L^2}{\Delta} \Rightarrow S_0 = 2\left(\dfrac{m^*\Delta}{2}\right)^{\frac{1}{2}} L = \sqrt{2}\,(m^*\Delta)^{\frac{1}{2}} L$

Anyway, we still have the tunneling amplitude
$e^{-S_0} \sim e^{-cL}$.

This behavior is different than the breaking of degeneracy due to finite-size effects that occurs in a system with conventional spontaneous symmetry breakdown:



Consider for example the (zero-temperature) Ising spin system on the torus. The state with all spins up tunnels to the state with all spins down as follows: An Island of flipped spins appears, then the island grows until it covers the whole torus. In the relativistic case, where the kinetic energy density of a moving domain wall is simply related to its tension σ

$$\text{amplitude} \sim \exp\left[-\sigma L_1 L_2\right]$$

– it falls off exponentially with the area of the torus, not its linear size.

Since the large gauge transformations U_1^q and U_2^q can not be built out of infinitesimal gauge transformations, the way they act on physical states is not determined by Gauss's law — All we can say is the the physical states transform as a representation of the $Z \times Z$ generated by the large gauge transformations.

The irreps are labeled by two angles, which (sorry!) I will call $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$

$$(U_1^q)^m (U_2^q)^n = e^{im\tilde{\Theta}_1} e^{in\tilde{\Theta}_2}$$

We can interpret e.g. $\tilde{\Theta}_1$ as the phase acquired by q vortices transported around the cycle C_2. Since $W(C_2) = e^{i\Theta_2}$ for an object of charge e, we should have

$$W_Q(C_2) = e^{i(Q/e)\Theta_2}$$

for an object of charge Q. A vortex with $\Phi = \Phi_0 = 2\pi/e$ has charge $Q = \mu\Phi_0 = \left(\frac{e^2}{2\pi} p/q\right)\frac{2\pi}{e} = e\, p/q$; so that

$$W_Q(C_2) = (e^{i\Theta_2})^{(p/q)}$$

so that

$$(U_1^q) = e^{ip\Theta_2}$$

For $p=1$, we can identify $\tilde{\Theta}_1 = \Theta_2$ (and $\tilde{\Theta}_2 = -\Theta_1$) — superselection sectors are labeled by the effect of transport of non-anyonic charged objects around cycles of the surface.

For nontrivial $\tilde{\Theta}$, there is an offset in the energy of the low lying excitations, because the quantum tunneling amplitude has a phase. E.g.

$$U_1 U_2 = U_2 U_1 e^{2\pi i\nu} = U_2 U_1 e^{2\pi i/q}$$

means that U_2 advances the phase of the eigenvalue of U_1 by $e^{2\pi i/q}$. But if one eigenvalue is $U_1 = e^{i\tilde{\Theta}_1/q}$, this means the other eigenvalues are

$$U_1 U_2^K |\tilde{\theta}_1/q\rangle = \exp[i(\tilde{\theta}_1 + 2\pi K)/q]\, U_2^K |\tilde{\theta}_1/q\rangle$$

($\tilde{\theta}_1$ is an angular variable defined modulo 2π, since $\tilde{\theta}_1 \to \tilde{\theta}_1 + 2\pi$ has the same effect as advancing $K \to K+1$; i.e., the spectrum of eigenvalues is left unchanged)

The effective tunneling Hamiltonian associated with U_1 (corresponding to a vortex tunneling around C_2) has a phase

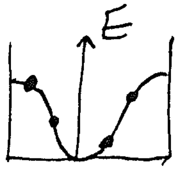
$$H|n\rangle = ce^{-S}\left(e^{i\tilde{\theta}_1}|n+1\rangle + e^{-i\tilde{\theta}_1}|n-1\rangle\right)$$

We again construct eigenstates $|K\rangle = \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} e^{-2\pi i K n/q} |n\rangle$,

but now

$$H|K\rangle = ce^{-S}\left[e^{i\tilde{\theta}_1} e^{2\pi i K/q} + e^{-i\tilde{\theta}_1} e^{-2\pi i K/q}\right]|K\rangle$$

or $$E_K = 2ce^{-S}\cos\left[\frac{2\pi K + \tilde{\theta}_1}{q}\right]$$



The energies slide like beads along the cosine curve, and the spectrum is unchanged by $\tilde{\theta}_1 \to \tilde{\theta}_1 + 2\pi$

We'll find again in other contexts that large gauge transformations can be represented by phases (with different phases associated with diffent superselection sectors), and that these phases can shift the vacuum energy. In the case of C-S theory, the $\tilde{\theta}$-dependence of the vacuum energy $\propto e^{-mL}$, and disappears in the $L \to \infty$ limit (which suppresses tunneling). In QCD, we'll see that what happens is more interesting!

Because the ground state degeneracy of an anyonic system (for genus > 0) has a topological origin, it is a very robust property. Generic perturbations cannot lift the degeneracy (in the $L \to \infty$ limit), if there is a mass gap.

In fact (as we'll discuss a bit later) the ground state degeneracy on the torus can be related to the Hall conductance; that this degeneracy must make discrete jumps explains the sharp steps between Hall plateaus. The topological degeneracy provides a criterion for distinguishing distinct bulk phases of matter.

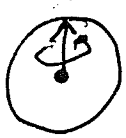
"Skyrmions" of the Nonlinear Sigma Model

Topological conservation laws, and Topologically stable defects can also be associated with spontaneously broken global symmetry

An example occurs for the global symmetry breaking pattern

$$G = SO(3) \to H = SO(2)$$ (E.g., Ferromagnet)

E.g. $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)^2 - \lambda/4(\phi^a\phi^a - v^2)^2$



the order parameter points in some direction, e.g. $\phi = v\begin{pmatrix}0\\0\\1\end{pmatrix}$

and the unbroken SO(2) symmetry preserves this axis

For λ large, it is useful to describe the theory by retaining only the manifold that minimizes the potential

Let $\phi^a = v n^a$ $\quad \begin{pmatrix}n^1\\n^2\\n^3\end{pmatrix}$ satisfies $n^a n^a = 1$

$$\mathcal{L} = \frac{v^2}{2}(\partial_\mu n^a)(\partial_\mu n^a) ; \quad \vec{n}^2 = 1$$

[A good description at energy << mass λv^2 of the scalar]

— effective description of the Goldstone excitations

Consider field configurations of finite energy; these satisfy

$$\nabla n^a \to 0 \text{ as } r \to \infty$$

In 2 spatial dimensions $\nabla n^a \to \frac{1}{r^{1+\varepsilon}}$ required by finiteness of $\int d^2x (\nabla n)^2$, Comment: each of the degenerate vacua $\langle vac| \phi^a |vac\rangle = v n^a$ lies in a distinct superselection sector: (78)

$\langle \hat{n} | \mathcal{O}_{local} | \hat{m}\rangle = 0$ for $\hat{m} \neq \hat{n}$

So $n \to$ constant (independent of θ) as $r \to \infty$. Hence — we can identify $r = \infty$ as a point

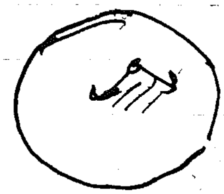
$n(r, \theta)$ is a map $S^2 \to S^2$

These maps have topological sectors labeled by an integer winding number.

Minimizing the energy in a nontrivial sector, we can find a nondissipative classical solution

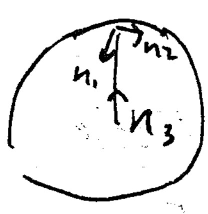
The winding number is

$$q = \frac{1}{4\pi} \int_{\substack{\text{Spatial} \\ S^2}} (\text{Volume element of "target" } S^2)$$



Since $d\vec{n} \perp \vec{n}$ on the target S^2

$$q = \frac{1}{4\pi} \int d^2x \; \varepsilon_{abc} \tfrac{1}{2} \varepsilon^{ij} n^a \partial_i n^b \partial_j n^c$$



This is volume element $dn^1 dn^2$ (in vicinity of $n^3 = 1$) written in rotationally invariant form

In form language: $q = \frac{1}{4\pi} \int \varepsilon_{abc} n^a \frac{1}{2} dn^b \wedge dn^c$

There is a bound on the energy in sector with topological charge q

Observe that

$$(\partial_i n^a \pm \varepsilon_{ij} \varepsilon^{abc} \partial^j n_b n_c)^2 = (\partial_i n^a)^2$$

$$\pm 2 \varepsilon^{ij} \varepsilon_{abc} \partial_i n^a \partial_j n^b n^c + \; (??)$$

We note that $\varepsilon_{ij}\varepsilon^{ik} = \delta_j{}^k$

$\varepsilon^{abc}\varepsilon_{ade} = \delta^b_d\delta^c_e - \delta^b_e\delta^c_d$

$$\Rightarrow (??) = \partial_j n^b \partial^j n^b n_c n^c - n^b \partial_j n^b n^c \partial^j n^c$$

$$2(\partial_i n^a)^2 \geq \mp 2\, \varepsilon^{ij}\varepsilon_{abc}\, n^a \partial_i n^b \partial_j n^c$$

0 because $\partial_j n^a n^a = 2 n^a \partial_j n^a = 0$

or $$\frac{v^2}{2}\int d^2x\, (\nabla\vec{n})^2 \geq \mp \frac{v^2}{2} 8\pi q$$

$$\Rightarrow E \geq 4\pi v^2 |q|$$

with equality for $\partial_i n^a = \varepsilon_{ij}\varepsilon^{abc}\partial^j n^b n^c$ $\quad q>0$

$-$ same $\quad q<0$

We can find configuration of minimal energy in each sector by solving 1st order eqns. (An exercise, and see below.)

We note that the solution, the skyrmion has an arbitrary scale, because the energy

$$E = \frac{v^2}{2}\int d^2x\, (\nabla\vec{n})^2$$

is scale invariant

$n^a(\vec{x})$ and $n^a(\lambda\vec{x})$

have the same energy — so if $n^a(\vec{x})$ is a solution, then so is $n^a(\lambda\vec{x})$

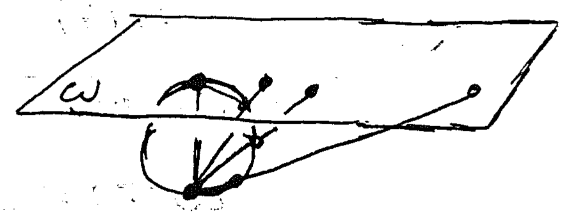
The (Multi)skyrmion solution

To solve the 1st order equation

$$\partial_i n^a = \varepsilon_{ij}\, \varepsilon^{abc}\, \partial^j n_b n_c \, ,$$

it is handy to represent the sphere $\{(n_1, n_2, n_3),\ \vec{n}^2 = 1\}$ via a stereographic projection that maps the sphere to the complex plane (Riemann sphere)

$$\omega = \frac{n_1 + i n_2}{1 + n_3}$$



The north pole ($n_3 = 1$) is mapped to the origin, the south pole to $\omega = \infty$

[Here the projected sphere actually has diameter 1

$$\frac{|\omega|^2}{1} = \frac{n_1^2 + n_2^2}{(1+n_3)^2} = \frac{1-n_3}{1+n_3}$$ from similar triangles ~]

We can show (an exercise) that the 1st order field equation is equivalent to (for $n > 0$)

$$\frac{\partial}{\partial \bar{z}}\, \omega(z, \bar{z}) = 0$$ where $z = x + iy$ is the spatial position on the plane

This is the Cauchy-Riemann condition -- the solution is an analytic (actually meromorphic) function

It is okay for the function to have poles, as these are just due to the coordinate singularity at $n_3 = -1$ (but branch points or essential singularities would be singularities in $n^a(\vec{x})$).

For $q < 0$, the Bogomolnyi eqn becomes $\partial_z \omega(z,\bar z) = 0$ and the solutions are antiholomorphic ($\omega = \omega(\bar z)$)

The solution $n^a(\vec x)$ approaches a constant at $r \to \infty$; by convention we may choose $\vec n \xrightarrow[r\to\infty]{} \hat e_z$; then $n_3(r=\infty) = 1$, and

$$\omega(z) \to 0 \quad \text{as} \quad z \to \infty$$

The simplest solution, then, is a single pole

$$\omega(z) = \frac{a}{z-b}$$

After shifting the origin so $b = 0$, and rotating axes so a is real and positive

$$\omega(z) = \frac{a}{z} \Rightarrow n_3 = \frac{1-|\omega|^2}{1+|\omega|^2} = \frac{1-\frac{a^2}{|z|^2}}{1+\frac{a^2}{|z|^2}} = \frac{r^2-a^2}{r^2+a^2}$$

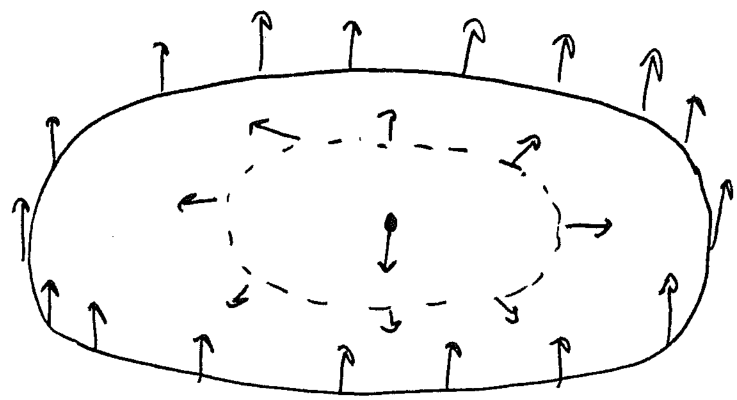
$$n_1 + i n_2 = \frac{2\omega}{1+|\omega|^2} = \frac{2a/z}{1+a^2/|z|^2} = \frac{2az^*}{|z|^2+a^2} = \frac{2are^{-i\theta}}{r^2+a^2}$$

($z = x+iy = re^{i\theta}$). So a is the size of the skyrmion, arbitrary because of the (classical) scale invariance. The general soln with one pole has 4 real parameters— Two for the position of the center (where $\vec n = -\hat e_3$), because of translation invariance, one for the size, because of scale invariance, and one for the orientation (the phase of a), because of rotational invariance.

[Comment: the skyrmion has rotational degeneracy, but the vortex does not— the effect of rotating the vortex is a gauge trans.]

(anti-)
The skyrmion:

$n_3 \to 1$ at $r = \infty$
$n_3 = -1$ at location of pole
(the "center")



$n_3 = 0$ where $w = 1$

The general solution with charge $|q| = n$ has n poles

$$w(z) = \frac{a_1 (z-a_2) \cdots (z-a_n)}{(z-b_1) \cdots (z-b_n)} \xrightarrow[|z|\to\infty]{} 0$$

The b_j's are the centers of the skyrmions -- the residues of the poles determine the size (modulus of residue) and orientation (phase of residue) of each skyrmion. The positions are arbitrary (no classical interactions among skyrmions). The n-skyrmion solution has $4n$ free real parameters. The antiskyrmion solution is obtained by replacing z by $\bar{z}$.

We see that the q is the winding number -- Since $w(z) = c$ is an nth-order polynomial equation, which has n roots: each point on the sphere is covered by $n = |q|$ points in space. An $z \to \bar{z}$ reverses orientation, while changing the sign of q

Lecture #9

The topological charge of the skyrmion can be expressed as the integral of a topological current:

$$J^\mu = \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\varepsilon_{abc}\, n^a \partial_\nu n^b \partial_\lambda n^c$$

and $q = \int d^2x\, J^0$ is conserved

because $\partial_\mu J^\mu = 0$:

$$\partial_\mu J^\mu = \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\varepsilon_{abc}\,\partial_\mu n^a \partial_\nu n^b \partial_\lambda n^c = 0$$

Why does it vanish? Because $\nabla\vec{n}$ spans the two-dimensional space orthogonal to $\vec{n}$; therefore, the triple vector product vanishes.

Sometimes it is handy to use the differential form notation. In three-dimensional (Euclidean) space, there is a volume form

$$dx^0\wedge dx^1\wedge dx^2 = \frac{1}{6}\varepsilon_{\mu\nu\lambda}dx^\mu dx^\nu dx^\lambda$$

that can be used to define the dual of a form
E.g.

$$dx^0 dx^1 dx^2 \leftrightarrow 1 \qquad (\text{3-form} \leftrightarrow \text{0-form})$$

$$dx^\mu \to \frac{1}{2}\varepsilon^{\mu\nu\lambda}dx_\nu dx_\lambda \qquad \begin{aligned} dx^0 &\to dx_1 dx_2 \\ dx^1 &\to dx_2 dx_0 \\ dx^2 &\to dx_0 dx_1 \end{aligned} \qquad (\text{1-form} \leftrightarrow \text{2-form})$$

$$dx_\mu dx_\nu \to \varepsilon_{\mu\nu\lambda}dx^\lambda$$

The 3-vector J_μ is associated with a 1-form whose dual is a 2-form

$$J = J_\mu dx^\mu$$
$$*J = \tfrac{1}{2}(*J)_{\mu\nu}\, dx^\mu dx^\nu$$

e.g. $J = \frac{1}{8\pi}\, \varepsilon_{abc}\, n^a \partial_1 n^b \partial_2 n^c dx^0 + \cdots$
$$= \frac{1}{8\pi}\, \varepsilon^{\mu\nu\lambda}\, \varepsilon_{abc}\, n^a \partial_\nu n^b \partial_\lambda n^c\, dx_\lambda$$
$$\Rightarrow *J = \frac{1}{8\pi}\, \varepsilon_{abc}\, n^a \partial_1 n^b \partial_2 n^c dx^1 dx^2 + \cdots$$
$$= \frac{1}{8\pi}\, \varepsilon_{abc}\, n^a \partial_\mu n^b \partial_\nu n^c dx^\mu dx^\nu$$

Associated with the map
$$\mathbb{R}^3 \longrightarrow S^2$$
defined by $(x^0, x^1, x^2) \longrightarrow (n^1, n^2, n^3)$,

and the volume form of S^2
$$V = \frac{1}{8\pi}\, \varepsilon_{abc}\, n^a dn^b \wedge dn^c,$$

is the "pull-back" of the volume 2-form to $\mathbb{R}^3$ – that is what $*J$ is

Now – since the volume V is a two form on a 2d space, its exterior derivative trivially vanishes
$$dV = 0$$
But $d*J$ is the pull-back of dV; hence
$$d*J = 0,$$
which is the current conservation eqn $\partial_\mu J^\mu = 0$

We can make our notation even more compact if we express $\hat{n}$, a (normalized) vector in the SU(2) Lie algebra as
$$\hat{n} = n^a(\tfrac{1}{2}\sigma^a), \quad \text{where } \operatorname{tr} \frac{\sigma^a}{2}\frac{\sigma^b}{2} = \frac{1}{2}\delta^{ab}$$
and use $\operatorname{tr}(\sigma^a \sigma^b \sigma^c) = 2i\varepsilon^{abc}$

$$\text{or} \quad \varepsilon^{abc} = -4i\, \mathrm{tr}\left(\frac{\sigma^a}{2}\frac{\sigma^b}{2}\frac{\sigma^c}{2}\right)$$

$$\Rightarrow V = \frac{-i}{\pi}\, \mathrm{tr}[n\, dn\, dn]$$

Now since $*J$ is closed,
and since there are no cohomologically nontrivial 2-forms in $\mathbb{R}^3$, it is also exact

$$*J = \frac{1}{2\pi} dA \qquad \text{(where } A \text{ is "globally defined")}$$

– in tensor notation

$$J^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda$$

– i.e. since J^μ has no divergence, it can be expressed as a curl – we have normalized it so that one unit of skyrmion topological charge corresponds to a quantum 2π of magnetic flux

$$q = \int d^2x\, J^0 = \frac{1}{2\pi}\int d^2x\, B = \frac{1}{2\pi}\oint A = \frac{1}{2\pi} 2\pi W = n$$

This relation of A to J arises as a field equation if we add to the Lagrange density a term

$$\mathcal{L}' = C\left[A_\mu J^\mu - \frac{1}{4\pi}\varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda\right]$$

Note that $S' = \int d^3x\, \mathcal{L}'$ has the gauge invariance

$$A_\mu \to A_\mu + \partial_\mu \Lambda,$$

if we can neglect surface terms,

since $\partial^\mu J_\mu = 0$ is an identity.

In terms of forms: $\mathcal{L}' = C\left[A * J - \frac{1}{4\pi} A\,dA\right]$

and S' is invariant under $A \to A + d\Lambda$, since $d * J = 0$

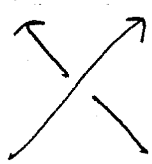
If we solve the field equation

$$J^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$$

and substitute back into the action

$$S' = \int d^3x\, \mathcal{L}' = C\int d^3x \left(\tfrac{1}{2} J^\mu A_\mu\right)$$

This is just as in our earlier discussion of abelian Chern-Simons theory, but now J^0 is the topological charge density, and $B = \text{curl}\,A$ is a magnetic flux associated with it. Thus when two skyrmions are exchanged, there



is an exchange phase (since $q = 1$)

$$\Theta \equiv C\,\tfrac{1}{2} q\,\underline{\Phi} = C\,\tfrac{1}{2}(1)\,2\pi = \pi C.$$

Thus, we identify $C = \frac{\Theta}{\pi}$ as the coupling constant in front of S', where Θ is the statistical angle of the identical skyrmions --- after solving the constraint relating J and A, we may write

$$S' = \int d^3x\,\mathcal{L}' = \frac{\Theta}{2\pi}\int d^3x\, J^\mu A_\mu$$

$$= \frac{\Theta}{4\pi^2}\int d^3x\, \varepsilon^{\mu\nu\lambda}\left(A_\mu \partial_\nu A_\lambda\right)$$

$$= \frac{\Theta}{4\pi^2}\int A\,dA$$

Consider Euclidean spacetime:

Now -- for a history of the $n^a(x)$ field that has finite <u>action</u> on $\mathbb{R}^3$, we must have

$$\vec{n}(\vec{x}) \longrightarrow \text{constant} \quad \text{as } |\vec{x}| \to \infty,$$

for
$$\int d^3x \; \nabla n \cdot \nabla n < \infty$$
(or vacuum $\to$ vacuum amplitude in infinite spatial volume..)
Thus we can identify $|\vec{x}| = \infty$ as a single point, as associate with the finite-action history a map

$$S^3 \longrightarrow S^2$$

where S^3 is spacetime, and S^2 is the "target space" $\{\vec{n}\}$ of the sigma model.

Furthermore, as long as skyrmions don't "cross" one another the term

$$S' = \frac{\theta}{4\pi^2} \int A\, dA$$

is a constant as we deform the history — it is a topological invariant.— What's going on — why do maps $S^3 \to S^2$ have a nontrivial topological classification?

Recall $dA = *J$ is the pull-back of the S^2 volume form to S^3, and that

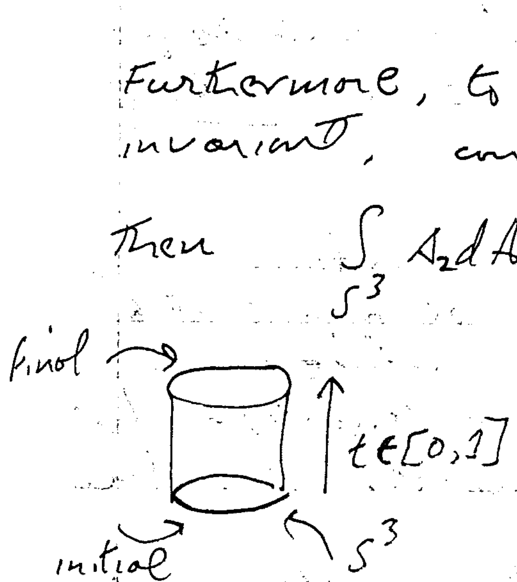
$\int A\, dA$ does not depend on how A such that $dA = *J$ is chosen, because

$$\int A\, dA \to \int (A + d\Lambda)\, dA$$

$$\Rightarrow \text{change is} \quad \Delta = \int_{S^3} d[\Lambda\, dA] = 0, \text{ by Stokes Thm}$$

Furthermore, to see that $\int A\,dA$ is a topological invariant, consider smoothly deforming A_1 to A_2

Then $\int_{S^3} A_2\,dA_2 - \int_{S^3} A_1\,dA_1 = \int_{S^3\times I} d[A\,dA]$



that is, let $t \in [0,1] \equiv I$ parametrize the deformation; then Stokes theorem implies

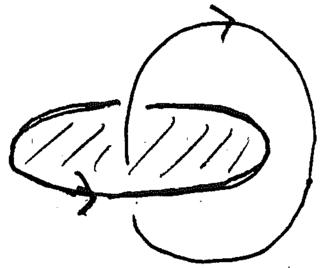
$$\Delta \int_{S^3} A\,dA = \int_{S^3\times I} dA\,dA$$

But $dA\,dA$ is just the pull back to $S^3\times I$ of the form $V\wedge V$ on S^2 — which vanishes because it is a 4-form on S^2; hence

$$\Delta \int A\,dA = 0$$ — it is a topological invariant, and so can change only if map $S^3 \to S^2$ has singularities.

This (integer-valued) invariant of a map $S^3 \to S^2$ is called the <u>Hopf invariant</u>, and it has a pretty geometrical interpretation. The inverse image of a point on S^2 is a circle (or sum of circles) in S^3; and if the map is nonsingular, the inverse images of two distinct points on S^2 are <u>nonintersecting</u> circles.

(Otherwise, the map would not be a function — a single point in S^3, where the circles intersect, would be mapped to two different points in S^2)

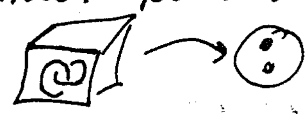


Two nonintersecting oriented loops in $\mathbb{R}^3$ have a linking number.

If one is the boundary of a nonselfintersecting orientable surface (not a knot), we define it by assigning an orientation to the surface via the right-hand rule, and counting the (signed) number of times that the other loop crosses the surface

what determines orientations of the circles? Frame normal to loop inherits orientation from S^2

For a map $S^3 \to S^2$, the linking number of the inverse images of two distinct points in S^2 is the Hopf invariant
(doesn't change as we vary the image points in S^2)



All physicists are familiar with the Hopf map, the standard map with unit Hopf invariant. It is the map of the normalized wave-function of a two-component spinor

$$\psi = e^{i\alpha}\begin{pmatrix} e^{-i\varphi/2}\cos\frac{\theta}{2} \\ e^{i\varphi/2}\sin\frac{\theta}{2} \end{pmatrix}$$

To

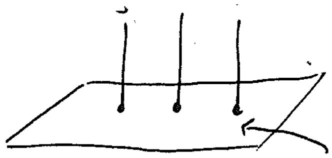
$$\psi^{+}\sigma^{a}\psi = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

The inverse image of $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ is the circle parametrized by α. As an exercise, you can check that

$$\psi_1(\alpha) = \begin{pmatrix} e^{i\alpha} \\ 0 \end{pmatrix}, \text{ the inverse image of } n_3 = 1$$

and $$\psi_2(\alpha) = \begin{pmatrix} 0 \\ e^{i\alpha} \end{pmatrix}, \text{ the inverse image of } n_3 = -1$$

have linking number 1 on S^3.

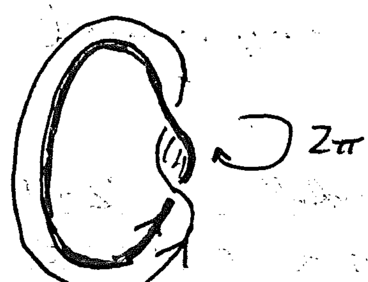


Also known as the "Hopf fibration" — we can view S^3 as a circle sitting atop each point of S^2

Identifying $S' = \Theta H$, where

$$H = \frac{1}{4\pi^2}\int A\,dA$$

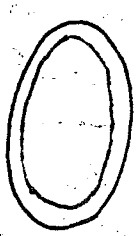
is the Hopf invariant, gives us another way to verify that the skyrmions are anyons with angle Θ.



To obtain a map $S^3 \to S^2$, we must consider a process with no skyrmions in the far past and future -- Suppose that a pair of skyrmions is created, one rotates by 2π, and then they reannihilate.

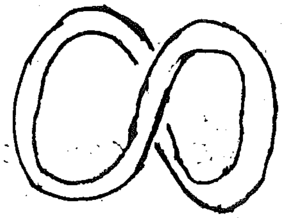
Follow in spacetime the worldlines of 2 points on the skyrmion, e.g. $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ and $\vec{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

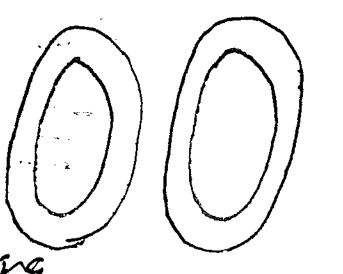
These two worldlines have linking number 1, while if there is no 2π rotation, the linking number is 0. Hence, the 2π rotation is weighted in the path integral by



$$e^{-2\pi i J} = e^{i\Theta H} = e^{i\Theta}$$

Similarly, consider creating two pairs, exchanging two skyrmions, and then reannihilating the pairs -- the world lines link; w/o the exchange, no linking

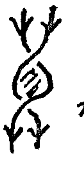




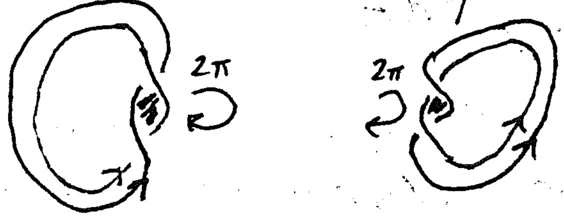
So -- the sector with the exchange is weighted by $e^{i\theta}$ relative to the sector with no exchange. Note that

Comment

becomes



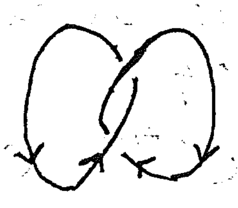
when you turn it upside down. It still twists the same way



it doesn't matter whether we rotate (or exchange) skyrmion or antiskyrmion, as long as we rotate (or exchange) with the same sense -- anyon and antianyon have the same angle

We see that the NL σ-model in 2+1 dimensions is an example of a classical theory that can be quantized in many different ways. Since the Hopf invariant H is a topological invariant, it makes no contribution to the classical field equations -- but the quantum theory is different for each $\theta \in [0, 2\pi)$, as θ determines the spin and statistics of skyrmions --

Another way to check the Hopf invariant: create two pairs and wind one skyrmion around another before the annihilate



This should be weighted by $e^{2i\theta}$

Since H is a topological invariant, we can deform to a configuration with "magnetic field" B isolated on the strings

$$\frac{\theta}{4\pi^2}\int d^3x\, A\cdot B = ?$$

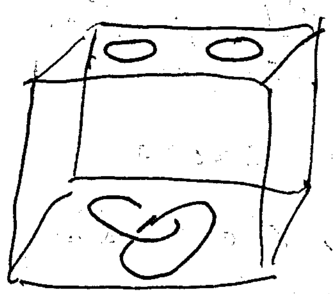
We have $\oint A = 2\pi$ along each string and $\int d^2x\, B = 2\pi$ across string

So we get contribution $4\pi^2$ integrating over each string or $8\pi^2$ total $\Rightarrow$ $e^{iS} = e^{2i\theta}$

Similarly, we may consider two strings of the abelian Higgs model in three spatial dimensions. Then the Chern-Simons term

$$L = \frac{e^2}{8\pi^2}\int d^3x\, \vec{A}\cdot\vec{B}$$ (now the flux quantum is $\Phi_0 = 2\pi/e$.)

is not really a topological invariant, but it is the linking number of the string loops up to a correction that is exponentially small if the string separation is large compared to the string thickness.



Consider a spacetime history of the strings in which the linking number changes

$$\Delta L = L_{final} - L_{init} = \int dt\, \frac{dL}{dt}$$

$$= \frac{e^2}{8\pi^2}\int d^4x\, \frac{\partial}{\partial t}\vec{A}\cdot\vec{B}$$

If we express this in covariant form

$$\Delta L = \frac{e^2}{8\pi^2}\int d^4x\, \partial_\mu\, \varepsilon^{\mu\nu\lambda\sigma} A_\nu \partial_\lambda A_\sigma$$

$$= \frac{e^2}{8\pi^2}\int d^4x\, \varepsilon^{\mu\nu\lambda\sigma}\partial_\mu A_\nu \partial_\lambda A_\sigma = \frac{e^2}{32\pi^2}\int d^4x\, \varepsilon^{\mu\nu\lambda\sigma} F_{\mu\nu}F_{\lambda\sigma}$$

$$= \frac{e^2}{16\pi^2}\int d^4x\, F_{\mu\nu}\tilde{F}^{\mu\nu} \quad \text{where } \tilde{F}^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$$

$$= \frac{e^2}{4\pi^2}\int d^4x\, \vec{E}\cdot\vec{B}$$

(SP)

In form language $L = \frac{e^2}{8\pi^2}\int A\,dA$

$$\Rightarrow \Delta L = \frac{e^2}{8\pi^2}\int F^2, \quad F = dA$$

$$= \tfrac{1}{2} F_{\mu\nu}dx^\mu dx^\nu$$

$$F^2 = \tfrac{1}{4} F_{\mu\nu}F_{\lambda\sigma}\,dx^\mu dx^\nu dx^\lambda dx^\sigma$$

$$= \tfrac{1}{4} F_{\mu\nu}F_{\lambda\sigma}\,\epsilon^{\mu\nu\lambda\sigma}dx^0 dx^1 dx^2 dx^3$$

$$= \tfrac{1}{2} F_{\mu\nu}\tilde{F}^{\mu\nu}d^4x$$

Now — suppose we include

$$\mathcal{L}' = \frac{e^2\theta}{4\pi^2}\vec{E}\cdot\vec{B}$$

in the Lagrangian of the abelian Higgs model
This is a total derivative

$$\mathcal{L}' = \frac{e^2\theta}{8\pi^2}\partial_\mu\left[\epsilon^{\mu\nu\lambda\sigma}A_\nu\partial_\lambda A_\sigma\right]$$

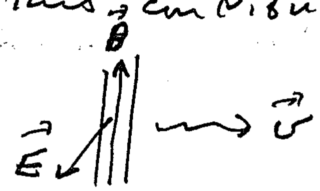
and so has no effect on the field equations
But it does enter the path integral
in processes in which strings cross

When strings cross, the linking number changes and

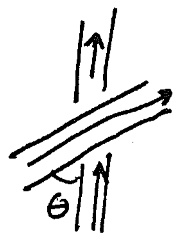
$$S' = \frac{e^2\theta}{4\pi^2}\int \vec{E}\cdot\vec{B}\,d^4x = \pm\theta$$

(with the sign depending on the sense of the crossing) and so the crossing is weighted by $e^{\pm i\theta}$

This contribution arises because a moving string



induces an electric field
$\vec{E} = \vec{v}\times\vec{B}$ perpendicular to string; and the crossing takes a time $\Delta t \sim R/v$
if R is the string thickness.



If they cross at an acute angle, the region of intersection is correspondingly larger — so we see that S' does not depend on the speed or angle of crossing

Note that the $e^{\pm i\theta}$ is a P and T violating phase that arises in scalar electrodynamics (the phase changes if we time reverse the crossing, or view it in a mirror) — but only in "nonperturbative" phenomena involving large gauge field strengths $\sim 1/e$

Nonabelian Strings

Lecture #10

The properties of vortices (or strings in three spatial dimensions) can be quite subtle when the unbroken group is nonabelian. We'll start by considering a particularly transparent example.

Consider a gauge theory with gauge group $G = SO(3)$, and a scalar field $\underline{\Phi}$ that transforms as the 5-dimensional irrep of $SO(3)$. Thus, $\underline{\Phi}$ can be represented as a real, symmetric, traceless 3×3 matrix. Under the gauge transformation $\Omega \in SO(3)$

$$\underline{\Phi} \to \Omega \underline{\Phi} \Omega^T$$

(where Ω^T is the transpose). Suppose that the Higgs potential $V(\underline{\Phi})$ is minimized by

$$\underline{\Phi} = \underline{\Phi}_0 \equiv v \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

— $\underline{\Phi}_0$ has two degenerate eigenvalues.

We can express $\underline{\Phi}_0$ as

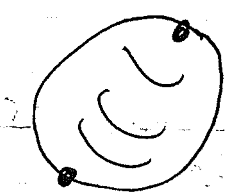
$$\underline{\Phi}_0 = v \left(I - 3 \hat{e}_3 \hat{e}_3^{\,T} \right)$$

where I is the 3×3 identity and $\hat{e}_3$ is the unit vector in the 3-direction. $SO(3)$ transformations (rotations) acting on $\hat{e}_3$ can take $\hat{e}_3$ to an arbitrary unit vector $\hat{e}$

Thus, the space of values of Φ that minimize $V(\Phi)$ (barring any accidental degeneracy not enforced by the gauge symmetry) is

$$\{\Phi = v(I - 3\hat{e}\hat{e}^T)\}.$$

We note that an inversion $\hat{e} \to -\hat{e}$ has no effect on Φ, so we can associate Φ with a unit vector, where $\hat{e}$ and $-\hat{e}$ are identified with the same value of Φ. Thus, the vacuum manifold is the two-sphere, but with antipodal points identified. This space is called RP^2, the real projective plane



(the space of undirected lines through the origin, in three-dimensional space).

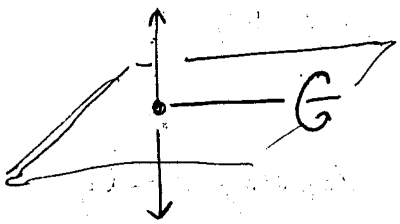
This manifold can also be expressed as G/H, where H is the subgroup of G that stabilizes Φ_0. The group of rotations that leave invariant an undirected vector pointing in the $\pm\hat{e}_3$ direction has two components. There is the

$SO(2)$ subgroup generated by

$$Q = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad - \text{rotations about } \hat{e}_3$$

$$SO(2) = \{ e^{i\theta Q},\ \theta \in [0, 2\pi) \}$$

And here is the (disconnected) component,



containing the $180°$ rotations about arbitrary axes in the x-y plane. For example, the $180°$ rotation about the x-axis is

$$\Omega_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$

an arbitrary element of the disconnected component is

$$\Omega_0 e^{iQ\theta}, \qquad \theta \in [0, 2\pi)$$

The full unbroken group is $O(2)$, the orthogonal group of the plane. We see that

$$G/H = SO(3)/O(2) = RP^2$$

$SO(3)$ is not simply connected, but has simply connected covering group $SU(2)$. An alternative description of the coset space is

$$G/H = SU(2)/Pin(2) = RP^2$$

$Pin(2)$ is the double cover of $O(2)$, with identity component

$$H_c = \{ e^{iQ\theta},\ 0 \le \theta \le 4\pi \} \qquad Q = \tfrac{1}{2}\sigma_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

and disconnected component

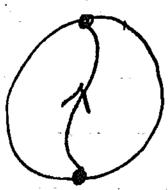
$$H_d = \{ X e^{iQ\theta} \quad 0 \le \theta \le 4\pi \}$$

where, e.g. $X = e^{i\pi/2\,\sigma_1} = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

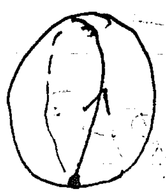
As abstract groups, $O(2)$ and $Pin(2)$ are actually isomorphic, differing just by a factor of 2 rescaling of θ.

This model contains topologically stable vortex solutions, because

$$\pi_1(G/H) = \pi_1(RP^2) = Z_2$$



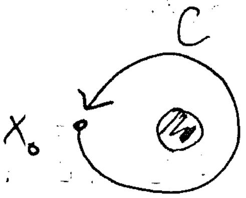
- A path on S^2 from point to antipodal point is a noncontractible closed path in RP^2

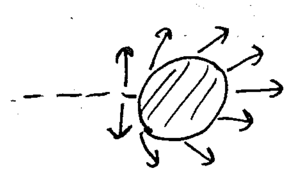


- Traversing the path twice, we obtain a contractible closed path on S^2

Since vortices are classified by Z_2, a vortex and antivortex are topologically equivalent.

If we represent an element of RP^2 as a unit vector (with the identification $\hat{e} \sim -\hat{e}$ understood), the behavior of the Higgs field outside the vortex core, in a particular gauge, looks like:





The effect of covariant transport in the gauge potential along a path C enclosing the vortex that starts and ends at the point X_0 is

$$U(C, x_0) = P\exp\left[i\oint_{C,x_0} A\right] = \Omega_0 e^{iQ\theta}$$

This transformation is in the disconnected component of $H(x_0)$, the group that stabilizes $\Phi(x_0)$

But because H is nonabelian, U is not invariant under the "unbroken" gauge transformations that preserve $\Phi(x_0)$; rather

$$U(C, x_0) \longrightarrow \Omega(x_0)\, U(C, x_0)\, \Omega(x_0)^{-1}$$

under a gauge transformation.

Look more closely at the structure of the unbroken group. We see that

$$\Omega_0 Q \Omega_0^{-1} = -Q$$

– a $180°$ rotation about x-axis followed by CCW rotation about z-axis followed by another $180°$ rotation about x-axis is equivalent to a CW rotation about z-axis. Therefore, the sign of an electric charge Q has no gauge-invariant meaning if the gauge group is $O(2)$... The irreps of $O(2)$ are two-dimensional, with eigenstates of Q with eigenvalues q and $-q$ combined in the same multiplet.

Since $\Omega_0 e^{iQ\theta}\Omega_0^{-1} = e^{-iQ\theta}$, we have

$$\left(e^{iQ\theta/2}\right)\left(\Omega_0 e^{iQ\theta}\right)\left(e^{-iQ\theta/2}\right) = \Omega_0 ;$$

that is -- there is a gauge transformation
$\Omega = e^{iQ\theta/2}$ under which

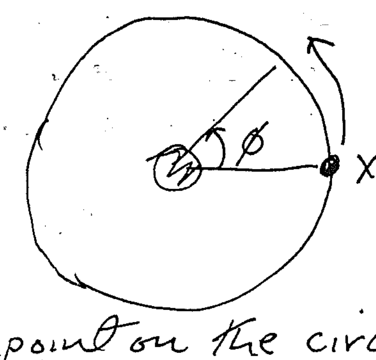
$$\Omega_0 e^{iQ\theta} \longrightarrow \Omega_0$$ — and so there is a gauge in which

$U(C, x_0) = \Omega_0$

is the effect of covariant gauge transport around the vortex.

The vortex of this model has a remarkable property: when an object with charge $Q = q$ is transported around the vortex, its charge flips to $Q = -q$. (Although the sign of q has no gauge invariant the relative charge of two objects does, so the charge flip has observable consequences... it has been called an "Alice vortex" because the effect of transport around the vortex is a reflection in the charge conjugation looking glass.

What is the mathematical meaning of the statement that Q changes sign? It is that the operator Q cannot be "globally defined" on a circle enclosing the vortex — we say that $SO(2)$ is "globally unrealizable" on the Alice vortex background.



If $\phi \in [0, 2\pi)$ parametrizes a point on the circle, the stabilizer group of the Higgs field at

ϕ is related to the stabilizer group at $\phi = 0$ by

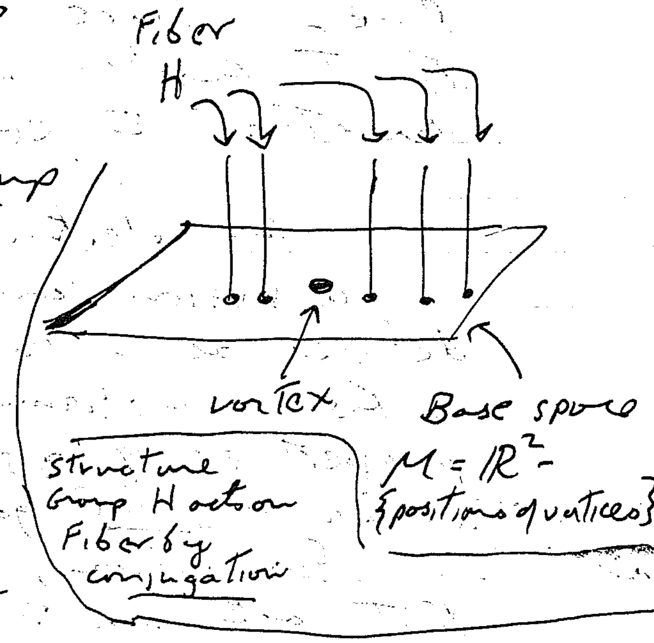
$$H(\phi) = U(\phi)\, H(0)\, U(\phi)^{-1}$$

where $$U(\phi) = P\exp\left(i\int_0^{\phi} d\phi\, A_\phi\right)$$

Of course $U(2\pi)\, H(0)\, U(2\pi)^{-1} = H(2\pi) = H(0)$,
but the generator Q satisfies

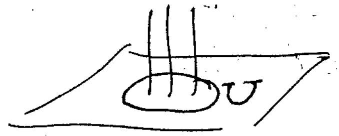
$$U(2\pi)\, Q\, U(2\pi)^{-1} = -Q\,,$$

because $U(2\pi) \in H_d$ — i.e. is in the disconnected component of the group



The stabilizer group of the condensate on the vortex background has the structure of a nontrivial (twisted) fiber bundle.

The base manifold of the bundle is the plane $\mathbb{R}^2$ with positions of vortices excised. The fiber sitting atop each position is the group H



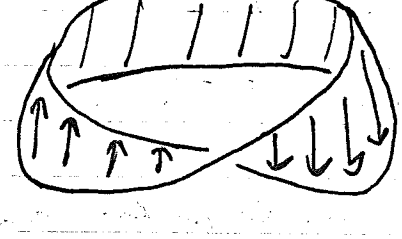
For each contractible open set $U \subset M$, the bundle has the "local" structure

$$U \times H$$

But the contractible open sets are glued together with a "twist," so that globally the bundle is not

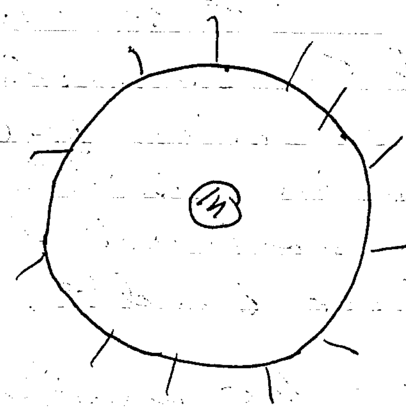
$$M \times H$$

This structure is closely analogous to that of a Möbius strip (or even more precisely, to a Klein bottle, a twisted product of circles).



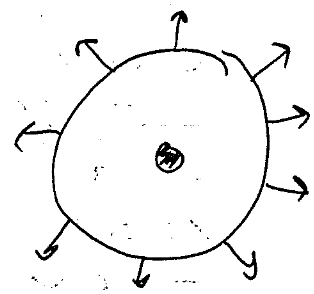
$H(\emptyset)$ corresponds to an undirected line through the origin in $\mathbb{R}^3$, and $Q(\emptyset)$ assigns a direction to the line — since the lines have a Möbius twist, there is no smooth way to assign a direction to each one

The electric charge can be defined in terms of the transformation properties of a state under a global gauge transformation, one that acts nontrivially at spatial infinity.



But if the world contains one (or an odd number of) Alice vortex (vortices), then global gauge transformation cannot be implemented, and the total electric charge is ill defined.

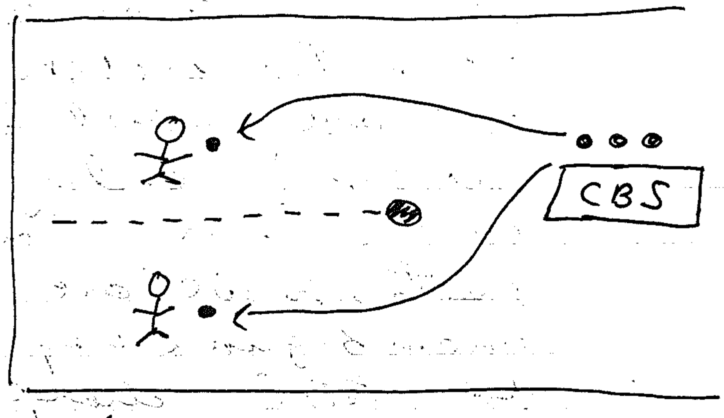
How does this pathology arise from a physical viewpoint? The electric charge inside a region can be measured from the boundary of a region, as the electric flux through the boundary



$$Q = \int d\vec{S} \cdot \vec{E}$$

and the an observer can measure $\vec{E}$ on the boundary by watching how test charges respond.

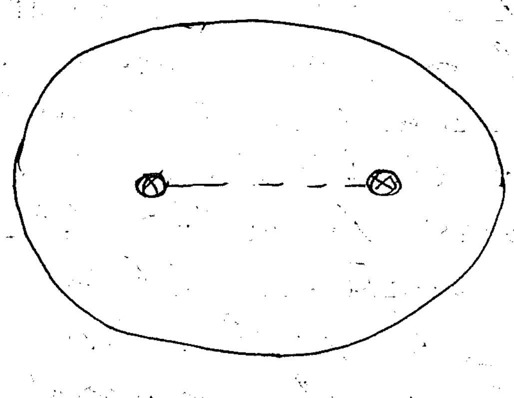
In the Alice world, the observer should be cautious, and carefully specify his convention for defining the sign of $\vec{E}$. So he establishes a



"charge bureau of standards" where the standard + and − charges are kept. The observer erects a curtain extending from the position of the Alice vortex to spatial ∞, and agrees that charges will never be permitted to cross the curtain. To measure the electric field at a point X, he carries a + charge from the CBS to X, never crossing the curtain. If this test charge is repelled, $\vec{E}$ points outward, and if it is attracted, $\vec{E}$ points inward.

But something is wrong — the $\vec{E}$ field measured by this procedure is discontinuous (changes sign) at the curtain. A test charge that is deemed to be + on one side of the curtain would be considered − on the other side. The discontinuity is an artifact of our conventions. If we allowed a test charge to pop through the curtain, the force exerted on it would be continuous. But if above the curtain we describe it as a + charge in an outward $\vec{E}$ field, then below the curtain we describe it as a − charge in an inward $\vec{E}$ field.

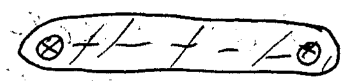
If we want the electric field to be smooth, we have to think of it as double-valued on the vortex background: it has two "sheets" that are glued together at a cut (the curtain) ending on the vortex. The electric "flux" has opposite signs on the two sheets, so we are left with no sensible unambiguous way to assign a total charge in the region. This is the physical origin of the unrealizability of the global gauge transformation.



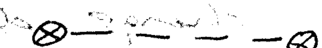
On the other hand, there is no obstruction to defining a global gauge transformation on a background of two Alice vortices, and correspondingly there is no obstacle to measuring the total charge. Now the curtain stretches from one vortex to the other, and there is no need for the observer to go near the curtain to carry out the flux measurement. True, the total charge still suffers from a sign ambiguity (of course, there are gauge transformations that flip Q), but at least now it is possible to establish a consistent convention. Technically, because of the Gauss law constraint, how a global gauge transformation acts on a state is determined only by its behavior at spatial infinity; we can choose it to be the identity near the vortices and the cut to avoid any potential singularities

Mathematically, we can choose a gauge such that

$$\underline{\Phi}(x) = \underline{\Phi}_0 = v\,\mathrm{diag}(1,1,-2)$$



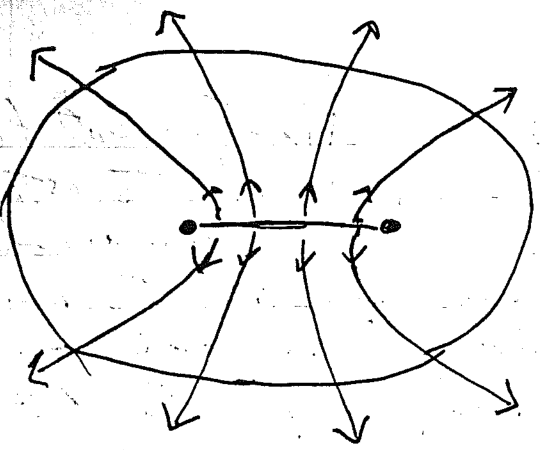
is constant everywhere outside the narrow shaded region, so that the order parameter does all of its twisting (traversal of a noncontractible loop in RP^2) in the shaded region. The shaded region corresponds to the curtain. Formally, it becomes in a suitable limit (="singular gauge") the cut in the plane where the electric field changes sign



In this singular gauge, we can solve the electromagnetic field equations in the two vortex background by imposing the boundary condition

$$\vec{E}(\text{just above cut}) = -\vec{E}(\text{just below cut})$$

These equations have a remarkable solution:



This solution has a nonzero electric charge — the electric flux through a surface around the vortex pair is nonvanishing

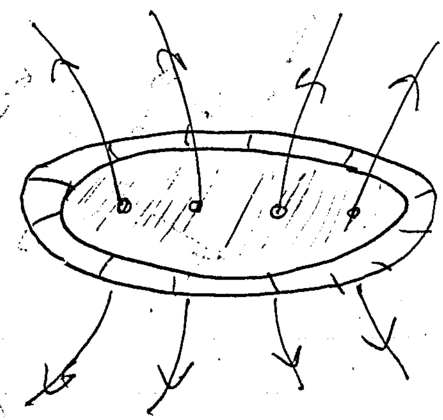
But there is no localized source for this electric field. The observer who detects the electric charge from afar goes on an expedition to determine from where the electric field lines emanate — but his quest is doomed to failure.

In our singular gauge description, the cut appears to be a source of electric charge. But the cut is an artifact of our conventions — there is no locally measureable charge density residing on the cut. If our test charge popped across the cut, the force exerted on it behaves continuously. The apparent charge density on the cut is an artifact, but there is no questioning the reality of the electric flux measured far away. In the presence of Alice vortices, electric charge need not have any localized source. Charge with no source has been called "Cheshire charge." (Cf a smile without a cat.)

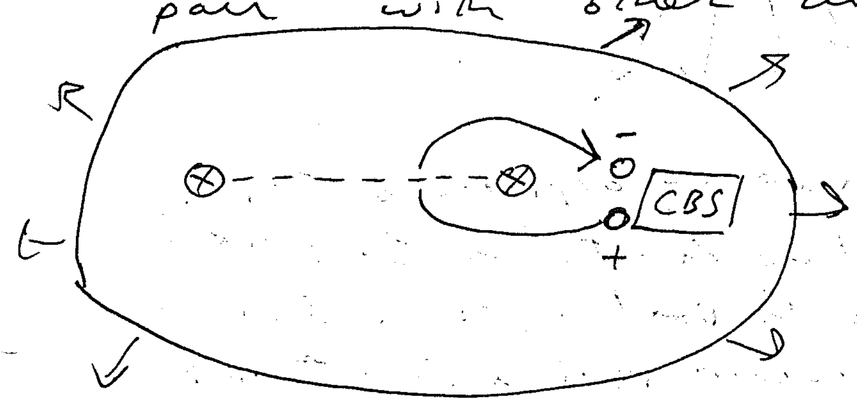
There is a technical flaw in the above discussion: In two spatial dimensions the self energy of a charge actually diverges logarithmically in the IR:

$$\int d^2x \left(\frac{Q}{2\pi r}\right)^2 = \frac{Q^2}{2\pi}\int \frac{dr}{r} \sim \frac{Q^2}{2\pi}\log R$$

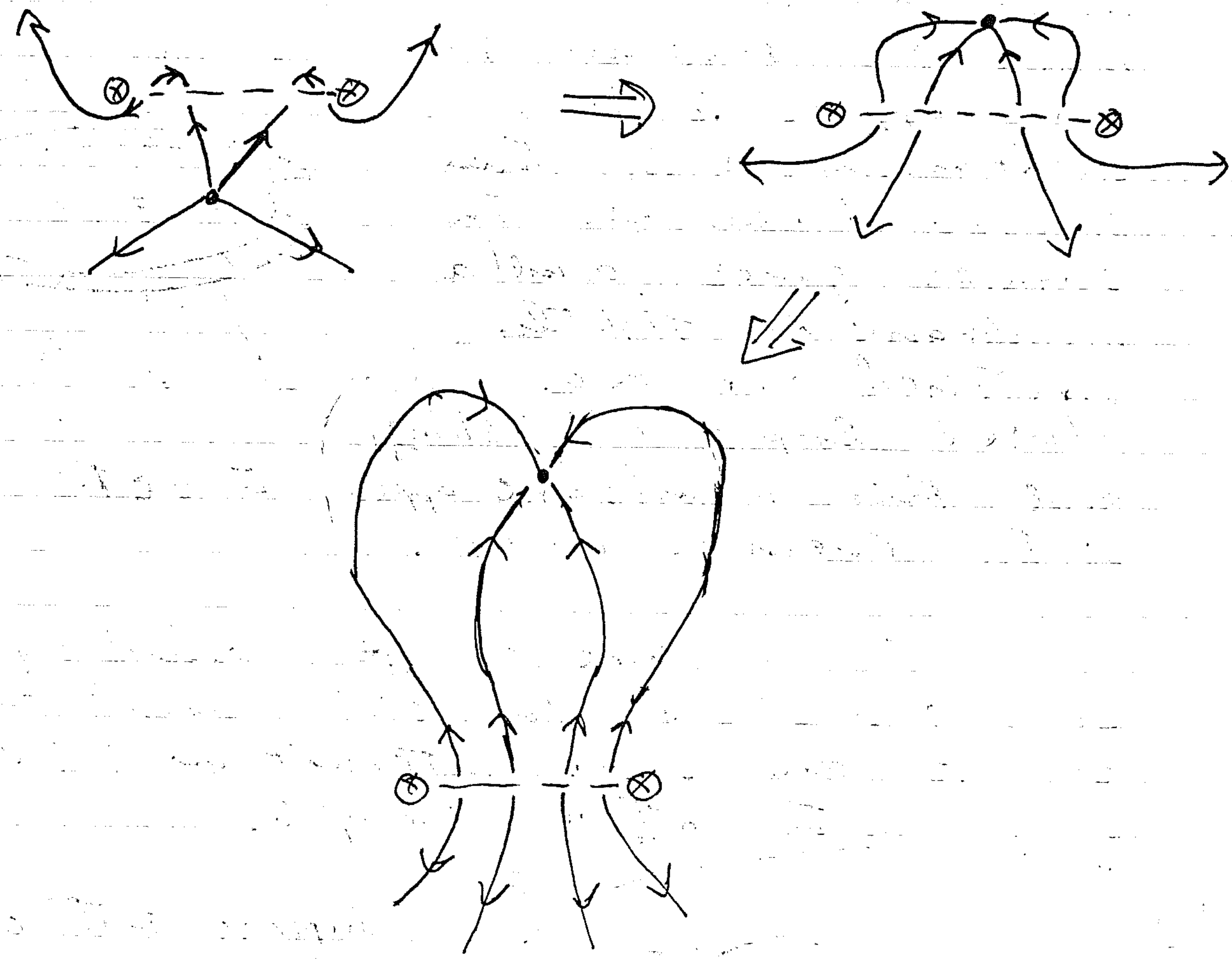
— there are no isolated charges because the Coulomb interaction is (logarithmically) confining in two dimensions. But much the same discussion also applies to Alice electrodynamics in three spatial dimensions. In singular gauge, a membrane is tightly stretched across a closed loop of Alice string, and this membrane appears to be the source of the Cheshire charge.



The Cheshire charge carried by a pair of vortices (or string loop) can change due to the interactions of the pair with other charges.



Suppose that charge that has been callibrated as q at the CBS is pulled through the pair and returned to the CBS. Now it is identified as $-q$. But this process cannot change the total charge of the world as determined by the flux through a distant surface. If +q changes to −q, there must be a compensating charge +2q that appears to make up the difference. Consider the singular-gauge description of how the electric field behaves as the charge passes the cut



As the point charge +q approaches the pair, the boundary conditions do not allow the electric field lines to cross the vortex, so the lines are bent back. The charge reaches the cut and disappears, just as its image charge on the second sheet appears from under the cut. As the charge (−q

Note: 

For a field line to slip past the vortex, charge would have to be created -- but all charged particles are massive, and we can execute the "pass through" adiabatically

pulls away, the flux emanating from it returns through the cut to the second sheet, while the flux emanating from the (+q) charge on the second sheet returns to the first sheet through the cut. When the point charge moves far away, the cut appears to be a source of charge +2q. Thus the total charge is conserved

$$(q)+(0) = (-q)+(2q),$$

but charge is transferred from point charge to vortex pair

The same concept — Cheshire charge carried by a loop of flux — resolves the "color conservation" problem described in Lecture #1.

As noted, the sign of q is actually gauge dependent, but we can describe the transfer of quantum numbers in a gauge invariant language. The irreducible representations of $O(2)$ are two dimensional and labeled by the nonnegative integer (or in $Pin(2)$ - half integer) $|q|$. The "fusion rules" for a direct product of representations say

$$|q_1| \otimes |q_2| = |q_1+q_2| \oplus |q_1-q_2|$$

In the initial configuration, the point charge is in irrep $|q|$, the vortex pair $|0|$ and the total charge is

$$|q| = |q| \otimes |0|$$

after the interaction, the point charge is still $|q|$, but the vortex pair is now $|2q|$ and the total charge is still

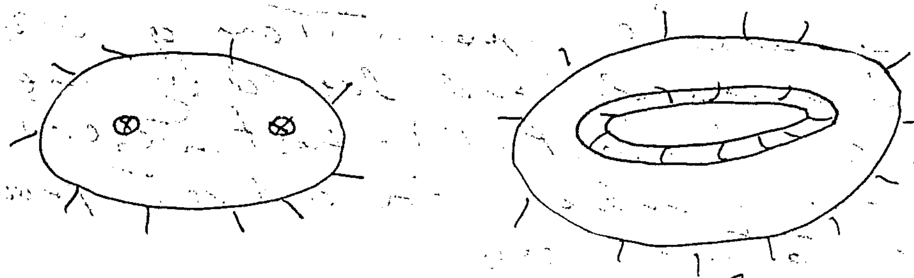
$$|q| \subset |q| \otimes |2q|$$

The irrep of the vortex pair changes while the irrep of the whole world does not, which eases any lingering suspicion that the charge transfer is a mere gauge artifact.

Lecture #11

So far, our discussion of Cheshire charge has been entirely classical. We'll need the quantum theory to see (for example) that Cheshire charge is quantized.

Formally, electric charge characterizes transformation properties under global gauge transformations.



We can consider a gauge transformation that is constant outside the region that contains a pair of Alice vortices, or a loop of Alice string in three dimensions.

Global transformations characterize the interactions of an object with another object that