Physics 230

Exercises
Due: Friday, 4 February 2000

1.1 Chromostatics

We saw in class that, in the one-gluon-exchange approximation, the static potential between two colored objects separated by distance r can be expressed as

$$E(r) = \frac{g^2}{4\pi r}\, T_1^a T_2^a \ ; \tag{1}$$

here $T^a_{1,2}$ are the $SU(3)$ generators acting on the objects, which have been normalized so that, in the defining (three-dimensional) irreducible representation of $SU(3)$,

$$\mathrm{tr}\ T^a T^b = \frac{1}{2}\delta^{ab} \ . \tag{2}$$

a) Two objects that transform as the (6) and (3) irreducible representations of $SU(3)$ can be combined in either the (10) or (8) representation. Find the static potential between two such objects in these two possible color channels. **Hint:** Under the SU(2) subgroup of $SU(3)$, embedded so that the (3) of $SU(3)$ transforms as $(2)+(1)$ under $SU(2)$, the (8) and (10) decompose as

$$\begin{aligned} (8) &\to (3)+(2)+(2)+(1)\ , &(3)\\ (10) &\to (4)+(3)+(2)+(1)\ . &(4)\\ & &(5) \end{aligned}$$

b) The combination of a (6) and $(\bar{3})$ can transform as a (3) or (15). Find the static potential in the two channels. **Hint:** The (15) decomposes under $SU(2)$ as

$$(15) \to (4)+(3)+(3)+(2)+(2)+(1)\ . \tag{6}$$

c) Now suppose that the static interaction energy of three colored objects at fixed positions is proportional to the group-theoretic factor

$$C = T_1^a T_2^a + T_2^a T_3^a + T_3^a T_1^a \ . \tag{7}$$

The direct product of three triplets decomposes into irreducible representations of $SU(3)$ according to

$$(3) \times (3) \times (3) \to (1) + (8) + (8) + (10) \ . \tag{8}$$

Evaluate the factor C in each of the possible channels.

1.2 From second to first order

a) Let ϕ be a charged complex scalar field in two spatial dimensions, and let $D_i = \partial_i - ieA_i$ denote the gauge-covariant derivative, where A_i is a $U(1)$ gauge potential. Show that

$$|D_1\phi \pm iD_2\phi|^2 = |D_1\phi|^2 + |D_2\phi|^2 \mp eB|\phi|^2 \pm i(\partial_1 J_2 - \partial_2 J_1) \ , \tag{9}$$

where $B = \partial_1 A_2 - \partial_2 A_1$ is the magnetic field, and $\vec{J} \equiv (J_1, J_2)$ is a certain vector current. What is $\vec{J}$?

b) Now consider the abelian Higgs model in two spatial dimensions, where the scalar self-coupling and the gauge coupling are related by $\lambda = e^2$ (the "Bogomol'nyi limit"). For static configurations with $D_0 = \vec{E} = 0$, the energy is therefore

$$E = \int d^2x \left[\frac{1}{2}B^2 + |D_1\phi|^2 + |D_2\phi|^2 + \frac{e^2}{2}(|\phi|^2 - v^2/2)^2 \right] . \tag{10}$$

Show that for any finite energy configuration,

$$E = \pm\pi v^2 n + \int d^2x \{ |D_1\phi \pm iD_2\phi|^2 \} + \frac{1}{2}\left[B \pm e(|\phi|^2 - v^2/2) \right]^2 \} \ , \tag{11}$$

where n is the winding number of the phase of the Higgs field at spatial infinity.

c) Find first-order field equations that are satisfied by the static solution for n vortices or n anti-vortices.

1.3 Gauge-invariant mass

Consider the free $U(1)$ gauge field theory in 2+1 spacetime dimensions, with Chern-Simons action

$$S = \int d^3x \left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\mu}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \right] \ . \tag{12}$$

a) Add a gauge fixing term

$$S_{\text{g.f.}} = \int d^3x \frac{-1}{2\alpha}(\partial^\mu A_\mu)^2 \ , \tag{13}$$

and then invert the quadratic form $\exp[i(S + S_{\text{g.f.}})]$ to find the gauge field propagator in momentum space. **Hint**: Defining

$$\begin{aligned} T_{\mu\nu} &= \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \ , && (14) \\ L_{\mu\nu} &= \frac{k_\mu k_\nu}{k^2} \ , && (15) \\ S_{\mu\nu} &= \frac{\epsilon_{\mu\nu\lambda} k^\lambda}{|k|} \ , && (16) \\ & && (17) \end{aligned}$$

you'll find it helpful to use the identities (in matrix notation)

$$\begin{aligned} TT = T \ , \quad LL = L \ , \quad TL = LT = 0 \ , && (18) \\ S^2 = -T \ , \quad TS = ST = S \ , \quad SL = LS = 0 \ . && (19) \\ && (20) \end{aligned}$$

b) Suppose that the gauge potential of the Chern-Simons theory is coupled to a conserved current, so that the term

$$S' = \int d^3x J^\mu A_\mu \tag{21}$$

is added to the action, where $\partial_\mu J^\mu = 0$. Consider the amplitude for a process in which a single gauge boson is exchanged between two conserved sources. Identity the poles in the amplitude, and interpret your result.