Physics 230

Exercises
Due: Friday, 25 February 2000

3.1 The skyrmion solution

We derived in class the first-order Bogomol'nyi equation satisfied by static solutions of the nonlinear sigma model in two spatial dimensions (for topological charge $q > 0$):

$$\partial_i n^a = \epsilon_{ij} \; \epsilon^{abc} \; \partial^j n^b \; n^c \; . \tag{1}$$

Express the spatial position (x_1, x_2) as a complex number $z = x_1 + i\,x_2$, and perform a stereographic projection to define

$$w(z, \bar{z}) = \frac{n_1 + i\; n_2}{1 + n_3} \; . \tag{2}$$

Show that the Bogomol'nyi equation is equivalent to the Cauchy-Riemann condition

$$\frac{\partial}{\partial \bar{z}} w(z, \bar{z}) = 0 \; . \tag{3}$$

3.2 The Hopf invariant and the linking number

In this exercise you will derive the connection between the Hopf invariant and the linking number that was used in class.

a) First we'll need a way to express the linking number of two directed loops in S^3 as an integral of a three-form. Denote the loops by C_1 and C_2, and imagine replacing each loop by a narrow tube containing a quantum of "magnetic flux," with the direction of the magnetic field determined by the orientation of the loop. That is, we associate with C_1 a two-form F_1 such that F_1 vanishes outside the tube, and the integral of F_1 is unity on a (suitably oriented) disk that crosses the tube. (In this exercise, we will use a different normalization than in class – we take the flux quantum to be unity rather than 2π.) Since F_1 is a closed form, and there are no cohomologically nontrivial two-forms in S^3, there is a globally defined one-form A_1 such that $F_1 = dA_1$. Similarly, a one-form A_2 and two-form $F_2 = dA_2$ may be associated with the directed loop C_2. Show that the linking number of the two loops is

$$L(C_1, C_2) = \int_{S^3} A_1 \wedge F_2 = \int_{S^3} A_2 \wedge F_1 \ . \tag{4}$$

b) Now consider a function $g : S^3 \to S^2$. The inverse image of a point in S^2 is a circle (or sum of circles) embedded in S^3. Consider a point p_1 on S^2, and imagine replacing the point by an infinitesimal disk centered at p_1. Associated with this disk is a two-form f_1 on S^2 that vanishes outside the disk, and such that

$$\int_{S^2} f_1 = 1 \ . \tag{5}$$

This two-form has a pullback to S^3 denoted $F_1 = g^*(f_1)$, and since F_1 is closed, it is $F_1 = dA_1$ for a globally defined one-form A_1. Show that the Hopf invariant of the map g can be expressed as

$$H[g] = \int_{S^3} A_1 \wedge F_1 \tag{6}$$

(apart from a factor of $(2\pi)^{-2}$ that arises because our normalization convention for the gauge field is different here than in class). **Hint**: In class we defined H by considering the pullback of a different two-form on S^2 – the volume form v. First show that $f_1 - v = db$ where b is a globally defined one-form on S^2; then if B is the pullback of b to S^3, $F_1 \wedge B$ vanishes because it is the pullback of a three-form $f_1 \wedge b$ on S^2.

[Similarly, if p_2 is another point on S^2, there is an associated two-form f_2 on S^2 such that $\int_{S^2} f_2 = 1$, with pullback $F_2 = dA_2$ on S_3; another expression for the Hopf invariant is $H[g] = \int_{S^3} A_2 \wedge F_2$.]

c) Now use (a) and (b) to show that the Hopf invariant of g is the same as the linking number of the inverse images of any two points p_1 and p_2 on the two-sphere. **Hint**: First show that $f_1 - f_2 = dc$, where c is a globally defined one-form on S^2; then if C is the pullback of c to S^3, $F_1 \wedge C$ vanishes because it is the pullback of a three-form $f_1 \wedge c$ on S^2.

3.3 Planar topological degeneracy

Consider a two-dimensional system with two types of quasi-particles: "magnetic" and "electric" excitations. Both types are bosons (the exchange phase is trivial when two magnetic or two electric excitations are interchanged). The electric and magnetic excitations are distinguishable from one another, but when an electric excitation is adiabatically transported around a magnetic one, the wave function acquires an Aharonov-Bohm phase $\exp(2\pi ip/q)$, where p and q are positive integers with no common factor. Both the magnetic and electric excitations can be pair-created and pair-annihilated.

a) If this system resides on the torus T^2, what is the generic degeneracy of the ground state in the limit of infinite volume? What if it resides on a closed orientable Riemann surface of genus g?

b) On a surface with boundary, the ground-state degeneracy may depend on the boundary conditions. Suppose that the system resides on a (large) annular ring in the plane. First suppose that the boundary is rigid, and that the electric and magnetic excitations are reflected elastically from both the inner and outer boundaries of the annulus. What is the generic ground-state degeneracy in this case?

c) Consider the annular ring as in (*b*). The boundaries of the ring are still rigid (there are no light excitations that propagate on the boundary), and magnetic excitations bounce off the boundaries as before. But now electric excitations can *appear or disappear* one at a time at the boundary. What is the generic ground-state degeneracy?