Physics 230

Exercises
Due: Friday, 28 April 2000

5.1 Is a dyon a boson?

a) The classical equation of motion for a particle with mass m and electric charge e moving in the magnetic field of a monopole with magnetic charge g is

$$m\ddot{\vec{r}} = eg\frac{\dot{\vec{r}} \times \hat{r}}{4\pi r^2} \; . \tag{1}$$

Use this equation to show that

$$\vec{J} \equiv m\vec{r} \times \dot{\vec{r}} - \frac{eg}{4\pi}\hat{r} \tag{2}$$

is a constant of the motion.

b) In quantum mechanics, the angular momentum operator for a charge-e particle on the background of a charge-g monopole is

$$\vec{J} = \vec{r} \times (\vec{p} - e\vec{A}) - \frac{eg}{4\pi}\hat{r} \; , \tag{3}$$

where $\vec{A}$ is the vector potential of the monopole. Find the commutation relations satisfied by J_1, J_2, J_3.

c) The monopole vector potential can be defined on two hemispherical patches, with a nontrivial matching condition at the equator. Consider the gauge in which

$$\begin{aligned} \vec{A} \cdot d\vec{r} &= \frac{g}{4\pi}(1 - \cos\theta)d\phi \; , \quad 0 \le \theta \le \pi/2 \; , \\ \vec{A} \cdot d\vec{r} &= \frac{-g}{4\pi}(1 + \cos\theta)d\phi \; , \quad \pi/2 \le \theta \le \pi \; . \end{aligned} \tag{4}$$

Show that in this gauge the component J_3 of angular momentum becomes

$$\begin{aligned} J_3 &= -i\frac{\partial}{\partial\phi} - \frac{eg}{4\pi} \; , \quad 0 \le \theta \le \pi/2 \; , \\ J_3 &= -i\frac{\partial}{\partial\phi} + \frac{eg}{4\pi} \; , \quad \pi/2 \le \theta \le \pi \; . \end{aligned} \tag{5}$$

d) How does the eigenvalue spectrum of J_3 depend on the value of $\frac{eg}{4\pi}$? [**Hint**: In this gauge, the eigenstates of J_3 must be single-valued on each hemisphere, and of course the components of angular momentum must provide a representation of the commutation relations found in (*b*).]

e) Consider a *composite dyon* comprised of a bosonic monopole with magnetic charge g bound to a bosonic particle with electric charge e. Find the effect on the two-dyon wave function of interchanging two identical dyons. Is the dyon a boson? Is your answer compatible with the expected connection between spin and statistics? [**Hint**: Each dyon must be covariantly transported in the gauge potential of the other. You answer will depend on the value of eg.]

5.2 Electroweak strings and monopoles

In the standard electroweak model with the pattern of gauge symmetry breakdown

$$G = [SU(2)_L \times U(1)_Y]/Z_2 \to U(1)_{EM} \ , \tag{6}$$

there is a massless photon that couples to the generator eQ, and a massive Z^0 boson that couples to the generator

$$e_Z Q_Z \equiv \frac{e}{\cos\theta \sin\theta}\left[T^3 - Q\sin^2\theta\right] \ ; \tag{7}$$

here T_3 is the diagonal $SU(2)_L$ generator and θ is the Weinberg angle (defined by $\tan\theta = g_1/g_2$, where g_1 is the $U(1)_Y$ gauge coupling and g_2 is the $SU(2)_L$ gauge coupling). In the limit $\sin\theta \to 1$, only $U(1)_Y$ is gauged, and the model becomes the "semilocal" model of Problem [4.2], which has a stable string solution provided that the Higgs is lighter than the Z^0. For $\sin\theta$ close to one but less than one, this "electroweak string" remains metastable, but it can end on an "electroweak magnetic monopole."

a) The Higgs condensate has $T_3 = 1/2$ and $Q = 0$. The requirement that the Higgs field is covariantly constant outside the core of the string imposes a condition on the "Z^0-flux" trapped inside the string. From this condition and the form of $e_Z Q_Z$, infer the Aharonov-Bohm phase acquired by an electrically charged particle that is transported around an electroweak string with minimal Z^0-flux.

b) Find the total Q_{EM} magnetic flux g that emanates from the end of a minimal electroweak string.

5.3 The real Berry phase

For a real (*i.e.*, time-reversal invariant) Hamiltonian, generic degeneracies are co-dimension two. Consider a family of 2×2 Hamiltonians depending on the real parameters x, y,

$$H(x,y) = \begin{pmatrix} y & x \\ x & -y \end{pmatrix} , \tag{8}$$

nondegenerate for all (x, y) except $(0, 0)$

a) Construct Berry's connection for the ground state of this family, on the punctured (x, y) (excluding the origin). **Hint**: The Berry connection is the one-form $A = i(\phi, d\phi)$, where $\phi(x, y)$ is the ground state of $H(x, y)$. Choose the phase of $\phi(x, y)$ so that ϕ is smooth on the punctured plane.

b) Evaluate the holonomy of the connection

$$\exp\left(i \oint_C A\right) \tag{9}$$

where C is a closed loop that winds once around the origin of the x-y plane.

c) Now let $H(\vec{\lambda})$ be a family of *real* Hamiltonians parametrized by $\vec{\lambda}$. Find a general expression for $\exp\left(i \oint_C A\right)$, where A is the Berry connection of the ground state, and C is a closed loop in $\vec{\lambda}$-space that avoids all points where the ground state is degenerate, as a sum of contributions from all of the degeneracies that cross a two-manifold Σ in $\vec{\lambda}$-space with $\partial\Sigma = C$.

5.4 A bounce for a hole in a wall

Domain walls that can be bounded by strings may be described by an effective Euclidean action (in four-dimensional Euclidean space time)

$$S_E = \sigma \int_M V + \kappa \int_\Sigma A \; ; \tag{10}$$

Here M is the world-volume of the wall, V is the volume form on M, Σ is the world-sheet of the string, and A is the area form on

Σ. A domain wall can decay via nucleation of a string loop, and if the dimensionless parameter $\kappa^3/\hbar\sigma^2$ is small, then the decay rate can be calculated in the semiclassical approximation. Find a "bounce" solution to the Euclidean equations of motion that dominates the path-integral expression for the decay rate, and compute the action of the bounce to estimate the decay rate. (The rate has the form $Ae^{-B/\hbar}$; you are being asked to find B and needn't worry about A.)