Physics 230

Exercises
Due: Friday, 5 May 2000

6.1 Spin manifolds

a) For what values of n is $CP^n = SU(n+1)/U(n)$ a spin manifold?

b) For what values of n is $SU(n)/SO(n)$ a spin manifold?

Hint: As for the examples discussed in class, in both cases it suffices to check whether smooth transport for spinors can be defined on the minimal noncontractible two-sphere contained in the manifold. The minimal two-sphere in a coset space G/H can be associated with a mapping from the two-dimensional disk into G that maps the boundary of the disk into H, where the mapping from the boundary into H is the minimal noncontractible loop in H. You need to determine whether the trivial holonomy acting on the tangent vectors associated with this loop can be lifted to a trivial holonomy acting on spinors. To get started, you should find how the "broken" G generators (those that are not generators of H) transform under H.

6.2 Maurer-Cartan integral invariants

Associated with a map g from the a manifold M to a compact Lie group G is a one-form $g^{-1}dg$ on M that takes values in the Lie algebra of G. Show that the integral

$$\int_\Sigma \mathrm{tr}[(g^{-1}dg)^k] \tag{1}$$

is a topological invariant of the map g, where Σ is any closed k-dimensional submanifold of M. (This is trivially true for even k, since the integrand vanishes in that case.)

Hint: Under left multiplication by an infinitesimal transformation $I + \varepsilon$, $g^{-1}dg$ transforms as

$$g^{-1}dg \to g^{-1}dg + g^{-1}d\varepsilon g = g^{-1}[dgg^{-1} + d\varepsilon]g\ . \tag{2}$$

It suffices to show that the change in $\mathrm{tr}[(g^{-1}dg)^k]$ to linear order in ε is an exact form.

6.3 Quantization of the Chern-Simons coupling

A is a connection one-form taking values in the (antihermitian) Lie algebra of the compact Lie group G; under a gauge transformation $\Omega \in G$ it transforms according to

$$A \to \Omega^{-1} A \Omega + \Omega^{-1} d\Omega = \Omega^{-1}[A + d\Omega\Omega^{-1}]\Omega \ . \tag{3}$$

The *Chern-Simons three-form* ω_3 is

$$\omega_3 \equiv \mathrm{tr}\left[AdA + \frac{2}{3}A^3\right] = \mathrm{tr}\left[AF - \frac{1}{3}A^3\right] \tag{4}$$

(where the curvature $F \equiv dA + A^2$ transforms as $F \to \Omega^{-1} F \Omega$). It has the property

$$d\omega_3 = \mathrm{tr}\ F^2 \ . \tag{5}$$

a) Show that under a gauge transformation $\Omega \in G$, the Chern-Simons three-form transforms as

$$\omega_3 \to \omega_3 - d[\mathrm{tr}(d\Omega\Omega^{-1}A)] - \frac{1}{3}\mathrm{tr}[(\Omega^{-1}d\Omega)^3] \ . \tag{6}$$

b) The *Chern-Simons action* on the three-sphere S_3 is

$$S_{CS} \equiv \mu \int_{S^3} \omega_3 \ . \tag{7}$$

Express the change in S_{CS} under the gauge transformation $\Omega \in G$ in terms of the invariant

$$n = \frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}[(\Omega^{-1}d\Omega)^3] \ , \tag{8}$$

which takes integer values. (The trace is taken in the fundamental representation of the gauge group, with generators normalized so that $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$; thus $n = 1$ for the $SU(2)$ gauge transformation with unit winding number on the three-sphere.) **Remark**: Remember that A may not be globally defined on S_3 – don't be too cavalier about ignoring an integral over S^3 of "$d\omega$," if ω is not a globally defined form.

c) We can quantize the Chern-Simons theory by integrating over gauge fields A on the manifold M, weighted by the phase factor $\exp(iS_{CS})$. But the quantization is consistent only if the factor $\exp(iS_{CS})$ is gauge invariant. (Otherwise, contributions from gauge equivalent configurations cancel one another, and all correlation functions of gauge-invariant operators are become the ill-defined formal expression 0/0.) Find a condition on the coupling μ that must be satisfied in order for consistent quantization to be possible.

d) An alternative way to quantize the theory on manifold M is to use the property $d\omega_3 = \text{tr } F^2$ to express the action as

$$S_{CS} = \mu \int_M \omega_3 = \mu \int_N d\omega_3 = \mu \int_N \text{tr } F^2 \ , \tag{9}$$

where N is a four-manifold whose boundary is M. This way, the integrand is gauge invariant. Still, an ambiguity can arise because the four manifold N is not unique. If N and N' are two different four manifolds that both are bounded by M, then

$$S_{CS} - S'_{CS} = \mu \int_{N-N'} \text{tr } F^2 \ , \tag{10}$$

where $N - N'$ is a closed four-manifold. Find a condition on μ that must be satisfied in order that $\exp(iS_{CS})$ be independent of how N is chosen.

6.4 Abelian Chern-Simons theory revisited.

Consider a $U(1)$ connection A, normalized so that the flux quantum is

$$i \int F = \Phi_0 = 2\pi \ . \tag{11}$$

a) In the abelian case, the integral over four-manifold M of the second Chern class is

$$\nu \equiv \frac{-1}{8\pi^2} \int_M F^2 \ . \tag{12}$$

This expression vanishes if M is a four-sphere, but can be nonzero for $M = S^2 \times S^2$ if both two-spheres carry nonzero magnetic flux. Suppose that the magnetic flux through the first two-sphere (which is independent of the position on the second two-sphere) is $m_1\Phi_0$, and the magnetic flux through the second two-sphere (which is independent of the position on the first two-sphere) is $m_2\Phi_0$. Evaluate ν.

b) The abelian Chern-Simons form is $\omega_3 = AdA$, and the Chern-Simons action on the three-manifold M is

$$S_{CS} = \frac{\mu}{2}\int_M AdA \ . \tag{13}$$

Since $\pi_3(U(1)) = 0$, no condition needs to be imposed on μ to quantize the theory on the three-sphere. But if the manifold M is not simply connected, the situation can be more interesting – there are "large" $U(1)$ gauge transformations that have a nontrivial winding around noncontractible cycles of M, and we need to consider whether S_{CS} transforms under such large gauge transformations.

Consider, for example $M = S^2 \times S^1$. If the magnetic flux on S^2 is $m\Phi_0 \neq 0$, then a gauge transformation with winding n around S^1 *does* modify the action, and so we obtain by insisting that $\exp(iS_{CS})$ be gauge invariant a restriction on the allowed values of m given the value of μ. This condition can be derived following the reasoning of [6.3c] above, but it is easy to make a factor of 2 mistake by that method, so we will prefer the method of [6.3d]. Therefore, we express the Chern-Simons action on $S^2 \times S^1$ as

$$S_{CS} = \frac{\mu}{2}\int_{S^2\times D^2} F^2 \ , \tag{14}$$

where D^2 is a disk whose boundary is S^1. If all integer values of the magnetic flux m on the two-sphere are allowed, show that a condition on μ must be satisfied for path integral quantization to be consistent. For other values of μ, find the values of m that are permitted. **Remark**: Compare with the result of Problem [2.3].

c) If we couple the Chern-Simons gauge field to a charge density, then charge q and flux Φ are related by

$$q = \mu\Phi \tag{15}$$

What does your answer to (*b*) imply about the total charge on the two-sphere?

6.5 Dirac quantization for branes

Consider p-branes (as in Problem [2.2]) that couple to the $(p+1)$-form A_{p+1} via the action

$$S = e_p \int_M A_{p+1} \ , \tag{16}$$

where M is the closed world-volume of the brane, and e_p is its charge. This action respects the gauge invariance

$$A_{p+1} \to A_{p+1} + d\omega_p \ , \tag{17}$$

where ω_p is a p-form. A_{p+1} need not be globally defined on M, but an alternative expression for the action is

$$S = e_p \int_N F_{p+2} \ , \tag{18}$$

where $F_{p+2} = dA_{p+1}$ is the corresponding field strength (a gauge-invariant quantity), and N is a $(p+2)$-manifold whose boundary is M. We can quantize the brane by integrating over closed world-volumes weighted by the phase e^{iS}, but the quantization is consistent only if the factor e^{iS} is independent of how we choose the manifold N with boundary M.

a) Show that the quantum theory is consistent only if the total flux Φ_{p+2} through any $(p+2)$-sphere

$$\Phi_{p+2} \equiv \int_{S^{p+2}} F_{p+2} \tag{19}$$

satisfies the condition

$$\Phi_{p+2} = \frac{2\pi}{e_{p+1}} \cdot n \ , \tag{20}$$

where n is an integer.

b) For the case $p = 1$ (strings), show that the total flux through *any* orientable closed 3-manifold Σ satisfies the quantization condition

$$\int_\Sigma F_3 = \frac{2\pi}{e_2} \cdot n \ . \tag{21}$$

Hint: Follow the steps of the corresponding argument given in class for 0-branes. Use a "good cover" of Σ, and introduce a two-form $A_{2,\alpha}$ that is globally defined in each (contractible) patch α of the cover, with a nontrivial matching condition at the double intersections $\alpha\beta$:

$$A_{2,\alpha} - A_{2,\beta} = d\omega_{1,\alpha\beta} \ . \tag{22}$$

The total flux is a sum over the patches:

$$\Phi_3 \equiv \int_\Sigma F_3 = \sum_\alpha \int_\alpha F_{3,\alpha} = \sum_\alpha \int_{\partial\alpha} A_{2,\alpha} \ . \tag{23}$$

Express the total flux as a sum of contributions associated with *quadruple* intersections $\alpha\beta\gamma\delta$, each of which must obey a quantization condition.