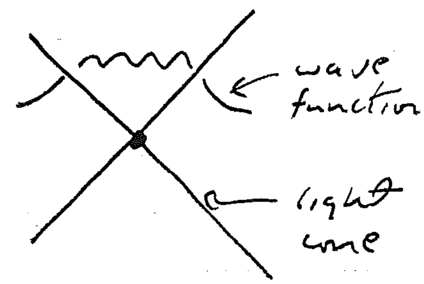
1. Quantum Field Theory on Flat Spacetime

We will consider how quantum theory is reconciled with *special* relativity. That is, we wish to construct a quantum theory such that

(i) physics is frame independent (Lorentz invariant),
(ii) relativistic causality is respected. (No propagation of information at $v > c$.)

We note to begin with that there is a certain tension between the principles of quantum theory and of relativity. The uncertainty principle causes a localized wave packet to spread quickly. What will prevent probability from leaking out of the light cone? (And thus, information would propagate backward in time, in some frames.) It will turn out that causality requires a rather subtle conspiracy.



We are accustomed to the notion that symmetries simplify physical problems. Thus, we might expect relativistic quantum theory — quantum theory constrained by Lorentz invariance — to be simpler than nonrelativistic quantum mechanics. This expectation turns out to be a bit too naive. The reason is that, in a relativistic theory, particle production is possible, and in fact, an indefinitely large number of particles can in principle be produced at sufficiently high energy. Therefore, relativistic quantum theory is inevitably a quantum theory with an infinite number of degrees of freedom. When we formulate perturbation theory, all possible states of the theory can appear as intermediate states. In a theory with an infinite number of degrees of freedom, we typically find --

- that the perturbation theory is complicated.
- perturbation theory suffers from (ultraviolet) divergences. Much of the subtlety of (relativistic) quantum field theory stems from these divergences; we need to understand their origin, and how to deal with them.

Particles:

We wish to construct a quantum theory of noninteracting relativistic particles. How shall we proceed. There are two basic strategies, complementary to one another.

1) Begin with a Hilbert space of relativistic particle states. Then introduce fields in order to construct observables that are localized in spacetime. This is called (for obscure reasons) "second quantization".

2) Begin with a relativistic (Lorentz invariant) classical field theory. Construct a quantum mechanical Hilbert space by elevating field to the status of an operator that obeys canonical commutation relation with its conjugate momentum. This is called "canonical quantization". One then finds relativistic particles in the spectrum of this theory.

Adopting either starting point, we are eventually led to the same theory. Strategy (1) is more logical and direct if our initial objective is to obtain a theory of particles. But (2) is reasonable if our goal is to construct a quantum theory with relativistically invariant (and causal) dynamics. Furthermore, when particles interact, the concept of a particle becomes ambiguous, and the advantages of (2) become more apparent.

Also — if spacetime is curved, the concept of a particle is again ambiguous, which leads one to favor (2). But we will see that (2) also suffers from ambiguities in curved spacetime. These ambiguities of (1) and (2) when spacetime is curved are central conceptual problems in the theory of quantum fields on curved spacetime.

We will describe procedure (1) in some detail, and then (2) somewhat more schematically. Then we will go on to try to apply these procedures to field theory on a nontrivial background.

Notation: we will usually set $\hbar = c = 1$. Our spacetime metric will be

$$\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$$

— following the convention used by Birrell and Davies

To a particle theorist defined on a flat background spacetime (like me) a relativistic particle is a unitary irreducible representation of the Poincare group. The Poincare group is the (semidirect) product of the Lorentz group and the translation group.

Lorentz transformation - $\Lambda : x^\mu \to \Lambda^\mu{}_\nu x^\nu$

$$\eta_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\sigma = \eta_{\lambda\sigma}$$

(We will always consider Λ to be a proper Lorentz transformation with $\Lambda^0{}_0 > 0$ and $\det \Lambda = 1$. These are Lorentz transformations that can be smoothly connected to the identity. That is, parity and time reversal are excluded.)

Translation - $a : x^\mu \to x^\mu + a^\mu$

The Poincare transformation
(Λ, a) acts as $(\Lambda, a) : x \to \Lambda x + a$

Thus, the composition law for Poincare transformation is

$$(\Lambda_1, a_1) \cdot (\Lambda_2, a_2) = (\Lambda_1 \Lambda_2, \Lambda_1 a_2 + a_1)$$

$$x \to \Lambda_2 x + a_2 \to \Lambda_1 \Lambda_2 x + \Lambda_1 a_2 + a_1$$

Now consider a quantum theory that respects Poincare invariance. It has a Hilbert space $\mathcal{H}$, and unitary operators that represent Poincare transformations acting on $\mathcal{H}$

$$(\Lambda, a): |\text{state}\rangle \longrightarrow U(\Lambda, a)|\text{state}\rangle$$

U is required to be unitary so as to preserve probability amplitudes

$$(\psi, \chi) = (U\psi, U\chi)$$

— this is what it means for Poincare transformations to be a symmetry.

Suppose we change reference frames twice. Then

$$|\text{state}\rangle \to U(\Lambda_1, a_1)\, U(\Lambda_2, a_2)\, |\text{state}\rangle$$

$$= U(\Lambda_1 \Lambda_2, \Lambda_1 a_2 + a_1)\, |\text{state}\rangle$$

is required for consistency. Thus

$$U(\Lambda_1, a_1)\, U(\Lambda_2, a_2) = U((\Lambda_1, a_1)\cdot(\Lambda_2, a_2))$$

— this is what it means to say that U's provide a representation of Poincaré group.

What is the structure of such a representation? First, consider translations $(\mathbb{1}, a)$. Translations are generated by the momentum operator

$$U(a) = e^{iP\cdot a}$$

The translation group is abelian, and its irreducible representations are one-dimensional. An IR acts on a state that is a simultaneous eigenstate of all components of P^μ

$$P^\mu |k\rangle = k^\mu |k\rangle$$

or

$$U(a)|k\rangle = e^{ik\cdot a}|k\rangle \quad \text{— "plane wave state"}$$

Obviously $U(a_1)U(a_2) = U(a_1 + a_2)$
– so this is a representation. (If we wish to think of k^μ as the four-momentum of a particle, note that $\hbar$ has already implicitly entered the discussion.)

How shall Lorentz transformations be represented? We'll consider the simplest case, in which a state with $\vec{k} = 0$ is invariant under rotations ("spin 0"). If we think of $|k\rangle$ as a particle with 4-momentum k^μ, then we expect

the state $U(\Lambda)\ |K\rangle$
to be a state with momentum ΛK
Indeed, this is required by the structure of the Poincare group

$$U(\Lambda^{-1})\,U(a)\,U(\Lambda) = U(\mathbb{1}, \Lambda^{-1} a)$$

or $$U(\Lambda^{-1})\, e^{iP\cdot a}\, U(\Lambda) = e^{iP\cdot(\Lambda^{-1}a)} = e^{i\Lambda P\cdot a}$$

$$\Rightarrow \quad U(\Lambda^{-1})\, P\, U(\Lambda) = \Lambda P$$

or $$P\,U(\Lambda) = U(\Lambda)\cdot(\Lambda P)$$, so that
$$P\,U(\Lambda)\,|K\rangle = (\Lambda K)\,U(\Lambda)\,|K\rangle$$

Our representation of the translation group becomes a representation of the Poincare group if we choose

$$U(\Lambda)\,|K\rangle = |\Lambda K\rangle$$ — up to normalization; to be discussed in a moment

The Lorentz transformations preserve the invariant

$$P^\mu P_\mu = m^2;$$

and for our representation to correspond to a physical particle, we demand

$$m^2 \geq 0 \quad \text{and} \quad P^0 \geq 0$$

then the states $|K\rangle$ with $K^2 = m^2$ and $K^0 > 0$ are the basis for an <u>irreducible</u> representation of the Poincare group. Any K on the mass hyperboloid can be obtained from e.g. $K = (m, \vec{0})$ by applying a suitable Lorentz transformation.

The relative normalization of the states $|K\rangle$ for various values of K can be determined from the requirement that $U(\Lambda)$ defined by

$$U(\Lambda)|K\rangle = |\Lambda K\rangle$$

is a unitary operator. Because the states $|K\rangle$ are a complete basis for the representation space, we have

$$\mathbb{1} = \int d\mu(K)\ |K\rangle\langle K|$$

for a suitable measure $d\mu$ defined on the hyperboloid. If U is unitary, then

$$\mathbb{1} = U(\Lambda)\,\mathbb{1}\,U(\Lambda)^{+} = \int d\mu(K)\ U(\Lambda)|K\rangle\langle K|U(\Lambda)$$

$$= \int d\mu(K)\ |\Lambda K\rangle\langle \Lambda K| = \int d\mu(\Lambda^{-1}K)\ |K\rangle\langle K|$$

Thus $d\mu(K)$ must be a <u>Lorentz invariant measure</u> satisfying

$$d\mu(K) = d\mu(\Lambda^{-1}K)$$

The invariant measure is unique up to an overall multiplicative factor and can be written as

$$d\mu(K) = \frac{d^4K}{(2\pi)^3}\ \delta(K^2 - m^2)\,\theta(K^0)$$

The measure d^4K is invariant because $\det \Lambda = 1$. Then $\delta(K^2-m^2)$ restricts this measure to the hyperboloid, and the $\Theta(K^0)$ further restricts it to the positive energy hyperboloid. The $(2\pi)^{-3}$ is a convention that fixes the overall normalization.

Since K^0 can be trivially integrated, we may also write

$$d\mu(K) = \frac{d^3K}{(2\pi)^3 2K^0} \text{ ; where } K^0 = +\sqrt{\vec{K}^2+m^2}$$

From $\quad \mathbb{I} = \int \frac{d^3K}{(2\pi)^3 2K^0} \, |K\rangle\langle K|$

and $\quad \mathbb{I}\,|K'\rangle = \int \frac{d^3K}{(2\pi)^3 2K^0} \, |K\rangle\langle K|K'\rangle = |K'\rangle$

we find the "relativistic normalization of states":

$$\langle K'|K\rangle = (2\pi)^3\, 2K^0\, \delta^3(\vec{K}'-\vec{K})$$

We have now completely specified the basis for the one-particle subspace $\mathcal{H}^{(1)}$ of Hilbert space in a theory of relativistic particles, and have defined the action of the Poincare group on this space. It acts irreducibly. The states in $\mathcal{H}^{(1)}$ are wave packets that can be expanded in terms of the plane wave basis

E.g.

$$|\tilde{f}(k)\rangle = \int \frac{d^3k}{(2\pi)^3 2k^0}\, \tilde{f}(k)\, |k\rangle$$

where $\quad U(\Lambda)|\tilde{f}(k)\rangle = \int \frac{d^3k}{(2\pi)^3 2k^0}\, \tilde{f}(k)\, |\Lambda k\rangle$

$$= |\tilde{f}(\Lambda^{-1}k)\rangle$$

and $\quad \langle \tilde{f}' | \tilde{f} \rangle = \int \frac{d^3k}{(2\pi)^3 2k^0}\, \tilde{f}'(k)^* \tilde{f}(k)$

(Note: There is a natural conjugate basis

$$|x\rangle = |\tilde{f} = e^{ik\cdot x}\rangle$$

such that $U(a)|x\rangle = |\tilde{f} = e^{ik\cdot(x+a)}\rangle = |x+a\rangle$

$\qquad U(\Lambda)|x\rangle = |\tilde{f} = e^{i(\Lambda^{-1}k)\cdot x}\rangle = |\Lambda x\rangle$

But in this basis, the inner product is

$$\langle x | x' \rangle = \int \frac{d^3k}{(2\pi)^3 2k^0}\, e^{-ik\cdot(x-x')} \text{ —}$$

a Lorentz invariant quantity that does not vanish for any value of $x-x'$. The significance of this will be discussed below.)

We have constructed a one-particle Hilbert space, but in anticipation of interactions that might change the number of particles (E.g. - measurements) we will enlarge it to a many-particle state.

First we need a vacuum - the zero-particle space. It is the unique state that is Poincare invariant:

$$P^\mu |0\rangle = 0 \qquad U(\Lambda)|0\rangle = 0$$

– the vacuum looks the same to all observers.

And we need many-particle states. We assume that the particles obey Bose statistics; e.g.,

$$|k_1, k_2\rangle = |k_2, k_1\rangle$$

so the normalization of the n-particle state $|k_1, k_2, \ldots k_n\rangle$ must respect the permutation symmetry acting on the n momenta. In the n-particle Hilbert space, the completeness relation becomes

$$(\mathbb{1})_{n\ \text{particle}} = \frac{1}{n!}\int \frac{d^3k_1}{(2\pi)^3 2k_1^0} \cdots \frac{d^3k_n}{(2\pi)^3 2k_n^0} |k_1, \ldots, k_n\rangle\langle k_1, \ldots, k_n|$$

where the $\frac{1}{n!}$ compensates for overcounting of states. The normalization is

$$\langle k_1 \ldots k_n | k_1' \ldots k_n'\rangle = n! \text{ terms}$$

e.g.

$$\langle k_1 k_2 | k_1' k_2'\rangle = \left((2\pi)^3\, 2k_1^0\, (2\pi)^3\, 2k_2^0\right) \times \left[\delta^3(\vec{k}_1 - \vec{k}_1')\,\delta^3(\vec{k}_2 - \vec{k}_2') + \delta^3(\vec{k}_1 - \vec{k}_2')\,\delta^3(\vec{k}_2 - \vec{k}_1')\right]$$

(In effect, we normalize states with coincident momenta differently than those with distinct momenta; this difference is compensated by the $1/n!$ in the sum over states.)

The many-particle states transform as the (reducible) representation of Poincare

$$U(\Lambda, a)\,|k_1, \ldots k_n\rangle = U(a)\,U(\Lambda)\,|k_1, \ldots k_n\rangle$$

$$= e^{i\Lambda(k_1 + \cdots + k_n)\cdot a}\,|\Lambda k_1, \ldots \Lambda k_n\rangle$$

(E.g., $(k_1+k_2)^2$ is Poincare invariant). And the full Hilbert space is a direct sum:

$$\mathcal{H} = \underbrace{\mathcal{H}^{(0)}}_{\text{0-particle}} \oplus \underbrace{\mathcal{H}^{(1)}}_{\text{1-particle}} \oplus \underbrace{\mathcal{H}^{(2)}}_{\text{2-particle}} \oplus \cdots$$

This (separable) Hilbert space is called <u>Fock space</u>

Now we have the complete Hilbert space of the theory, and a representation of the Poincare group acting on the space. Have we therefore completed the construction? No. This theory, so far, lacks <u>local observables</u> — operators with compact support in spacetime that are in principle measurable. Such observables are called <u>fields</u>. A field technically is an "operator-valued distribution" — i.e. if $\phi(x)$ is a field, then

$\int d^4x\, \phi(x) f(x)$ is an operator $\mathcal{H} \to \mathcal{H}$,

where f is a suitably smooth test function.

The idea of field theory is that quantities that can be measured by an observer localized in spacetime can be modeled as functions of the "smeared" fields.

Observers can communicate by emitting and absorbing particles, so the fields should be able to create or destroy particles — as operators, they mix up the different n-particle spaces $\mathcal{H}^{(n)}$.

A further motivation for introducing fields arises when we consider introducing interactions among particles. In a causal theory, particle interactions are local in spacetime, and admit a natural description in the field language.

We will construct a fundamental field of the theory from which all local observables can be constructed. The field $\phi(x)$ is to be regarded as a Heisenberg picture operator in the theory. (Since fields will depend on $\vec{x}$, it is natural that they depend on t as well, in a relativistic theory.) We'll demand that $\phi(x)$ have the following properties --

~~$\phi(x): \mathcal{H}^{(n)} \to \mathcal{H}^{(n-1)} \oplus \mathcal{H}^{(n+1)}$~~

~~$\mathcal{H}^{(0)} \to \mathcal{H}^{(1)}$~~

(i) ϕ creates or destroys one particle. Then operators that create or destroy many particles are obtained from polynomials in ϕ

(ii) $\phi = \phi^{+}$ — we want ϕ to be hermitian, so that it will be an observable

(iii) $U(\Lambda, a)\phi(x)U(\Lambda, a)^{-1} = \phi(\Lambda x + a)$

ϕ transforms as a scalar under Poincare transformations. — It is convenient to formulate a relativistic theory in terms of fields that transform as simply as possible

These conditions suffice to fix $\phi(x)$ almost completely.

To construct $\phi(x)$ it is useful to first define operators with momentum space arguments that create and destroy particles. Define an operator

$$A(k)^{+} : \mathcal{H}^{(0)} \to \mathcal{H}^{(1)}$$

by

$$A(k)^{+}|0\rangle = |k\rangle$$

Similarly, we define the action of $A(k)$ on $\mathcal{H}^{(n)}$ by

$$A(k)^{+}|k_1, \dots, k_n\rangle = |k, k_1, k_2, \dots k_n\rangle$$

— this determines all matrix elements of $A(k)^{+}$ between states in the Fock space $\mathcal{H}$, and hence also determines all matrix elements of its adjoint $A(k)$. For example —

$\langle 0|A(k)^\dagger|\text{arbitrary state}\rangle = 0 \Rightarrow A(k)|0\rangle = 0$

$\langle k'|A(k)^\dagger|0\rangle = \langle k'|k\rangle = (2\pi)^3(2k^0)\,\delta^3(\vec{k}-\vec{k}')$

$\Rightarrow A(k)|k'\rangle = (2\pi)^3(2k^0)\,\delta^3(\vec{k}-\vec{k}')|0\rangle$

Note that
$\langle 0|[A(k), A(k')^\dagger]|0\rangle = \langle k|k'\rangle = (2\pi)^3 2k^0\delta^3(\vec{k}-\vec{k}')\langle 0|0\rangle$

and one can check that
$\langle \text{state}'|[A(k), A(k')^\dagger]|\text{state}\rangle = (2\pi)^3 2k^0\delta^3(\vec{k}-\vec{k}')\langle \text{state}'|\text{state}\rangle$

– for arbitrary Fock space basis states

So $[A(k), A(k')^\dagger] = (2\pi)^3 2k^0\delta^3(\vec{k}-\vec{k}')$
is an operator identity in Fock space.

Note also that

$|k_1, \dots k_n\rangle = A(k_1)^\dagger \cdots A(k_n)^\dagger|0\rangle$

Thus – the assumed Bose symmetry of the many particle states implies

$[A(k)^\dagger, A(k')^\dagger] = 0.$

and hence $[A(k), A(k')] = 0$

How do the $A(k)$ transform under Poincare transformations? Using Poincare invariance of the vacuum —

$U(a)|k\rangle = e^{ik\cdot a}|k\rangle = U(a)A(k)^\dagger U(a)^{-1}U(a)|0\rangle$

$= U(a)A(k)^\dagger U(a)^{-1}|0\rangle = e^{ik\cdot a}A(k)^\dagger|0\rangle$

so $\quad U(a)\, A(k)^\dagger\, U(a)^{-1} = e^{ik\cdot a}\, A(k)^\dagger$

acting on vacuum, and similar acting on any state

take adjoint: $\quad U(a)\, A(k)\, U(a)^{-1} = e^{-ik\cdot a} A(k)$

$U(\Lambda)\,|k\rangle = |\Lambda k\rangle$, or

$U(\Lambda) A(k)^\dagger U(\Lambda)^{-1} |0\rangle = A(\Lambda k)^\dagger |0\rangle$

$$\Rightarrow \qquad U(\Lambda)\, A(k)^\dagger\, U(\Lambda)^{-1} = A(\Lambda k)^\dagger$$
$$U(\Lambda)\, A(k)\, U(\Lambda)^{-1} = A(\Lambda k)$$

Now, the most general operator that creates or destroys one particle is a linear combination of A's and $A^\dagger$'s:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2k^0} \left[C(k,x)\, A(k) + C(k,x)^* A(k)^\dagger \right]$$

transforms under Lorentz transformations as

$$U(\Lambda)\,\phi(x)\,U(\Lambda)^{-1} = \int \frac{d^3k}{(2\pi)^3 2k^0} \left[C(k,x)\, A(\Lambda k) + C(k,x)^* A(\Lambda k)^\dagger \right]$$

use Lorentz-invariance of measure

$$= \int \frac{d^3k}{(2\pi)^3 2k^0} \left[C(\Lambda^{-1}k,x)\, A(k) + C(\Lambda^{-1}k,x)^* A(k)^\dagger \right]$$

And we require

$$= \phi(\Lambda x) = \int \frac{d^3k}{(2\pi)^3 2k^0} \left[C(k, \Lambda x)\, A(k) + C(k,\Lambda x)^* A(k)^\dagger \right]$$

Therefore — $C(k,x)$ must be a Lorentz invariant function

$$C(\Lambda^{-1}k, x) = C(k, \Lambda x) \quad \text{or} \quad C(\Lambda k, \Lambda x) = C(k,x)$$

C is a function of the Lorentz-invariant variables $m^2 = k^2, x^2, k\cdot x$

Now consider translations --

$$U(a)\,\phi(x)\,U(a)^{-1} = \int \frac{d^3k}{(2\pi)^3 2k^0}\left[C(k,x)e^{-ik\cdot a}A(k) + C(k,x)^* e^{ik\cdot a}A(k)^\dagger\right]$$

$$= \phi(x+a) = \int \frac{d^3k}{(2\pi)^3 2k^0}\left[C(k,x+a)A(k) + C(k,x+a)^* A(k)^\dagger\right]$$

So C must satisfy

$$C(k, x+a) = e^{-ik\cdot a}\, C(k,x)$$

— this determines the x dependence up to a multiplicative constant

$$C(k,x) = C e^{-ik\cdot x}$$

In fact, the phase of C is unphysical — we can absorb the phase by adjusting the phase of $A(k)$, or equivalently by adopting a different phase convention for the states $|k\rangle$. Then we can take C to be real and

$$\phi(x) = C\int \frac{d^3k}{(2\pi)^3 2k^0}\left[e^{-ik\cdot x}A(k) + e^{ik\cdot x}A(k)^\dagger\right]$$

So we have determined ϕ up to a real normalization constant. This constant quantifies the amplitude for ϕ to create a one-particle state:

$$\langle k | \phi(x) | 0 \rangle = C e^{ik\cdot x}$$

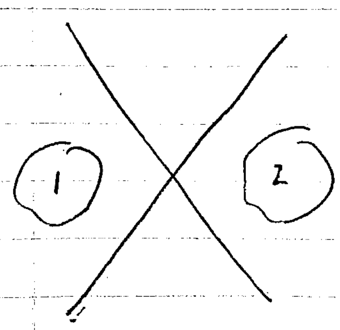
– We can choose $C=1$ by convention

The states $\phi(x)|0\rangle$ provide a basis for $\mathcal{H}^{(1)}$ that is in a sense conjugate to the plane wave basis. Note that this is a complete basis. And the

$$\phi(x_1) \cdots \phi(x_n) |0\rangle$$

are a complete basis for the n-particle space $\mathcal{H}^{(n)}$ – thus polynomials in the fields, acting on the vacuum, span the Fock space $\mathcal{H}$. We say, in this case, that the fields are a complete set of local observables.

Now we have a relativistic field theory with local observables, but there is one more thing to check – that the theory satisfies relativistic causality.



Causality requires that if measurements are performed in regions ① and ② of spacetime that have spacelike separation, the measurement in ① has no effect on outcome of measurement in ② – and vice-versa.

In quantum mechanics, this means that the observables measured in ① and ② must commute

$$0 = [\mathcal{O}(1), \mathcal{O}(2)] \qquad \text{① and ② spacelike separated.}$$

If all observables can be constructed from smeared fields, then it is necessary (and sufficient) that

$$[\phi(x), \phi(y)] = 0 \quad , \quad (x-y)^2 < 0 .$$

Is this true? Yes. But for a subtle reason.

Let's decompose $\phi(x)$ into a piece that annihilates particles and a piece that creates particles.

$$\phi(x) = \phi^{(-)} + \phi^{(+)}$$

$$\phi^{(-)} = \int \frac{d^3k}{(2\pi)^3 2k^0} e^{-ik\cdot x} A(k)$$

$$\phi^{(+)} = \phi^{(-)\dagger}$$

Consider the function

$$G_+(x-y) = [\phi^{(-)}(x), \phi^{(+)}(y)] = \langle 0 | \phi^{(-)}(x) \phi^{(+)}(y) | 0 \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3 2k^0} e^{-ik(x-y)}$$

Note that, because of the Lorentz invariance of the measure, G_+ is a Lorentz invariant function

$$G_+(\Lambda x) = G_+(x)$$

We could evaluate this integral (and express G_+ in terms of a modified Bessel function) but even without evaluating it explicitly, we can see that $G_+(x)$ does not have the property of vanishing for spacelike x. The reason is that G_+ is an analytic function, and it cannot vanish in an open set without vanishing throughout its domain of analyticity.

G_+ is analytic because of the positivity condition on the energy of a particle, $k^0 > 0$. If we express

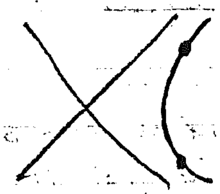
$$G_+(x) = \int \frac{d^4k}{(2\pi)^3} \, \theta(k^0) \, \delta(k^2 - m^2) \, e^{-ik\cdot x}$$

we see that G_+ is a sum of entire functions $e^{-ik\cdot x}$, and because k^0 is restricted to be positive, this sum converges for $\mathrm{Im}\, x^0 < 0$ — it is damped by the exponential. So Real x^0 is at the very least on the boundary of the domain of analyticity. In fact, G_+ does decay like $\exp[-m|x^2|^{\frac{1}{2}}]$ for x outside the light cone, but it is not zero.

This sounds like a serious breach of causality — since $G_+(x) = \langle 0|\phi^{(-)}(x)\,\phi^{(+)}(0)|0\rangle \neq 0$ for x spacelike, the particle excitation localized at x seems to propagate outside the light cone, just as we feared. But can this propagation really be detected?

Let's consider a commutator of observables:

$$iG(x-y) = [\phi(x), \phi(y)] = [\phi^{(-)}(x), \phi^{(+)}(y)] + [\phi^{(+)}(x), \phi^{(-)}(y)].$$
$$= G_+(x-y) - G_+(y-x)$$



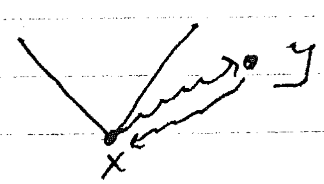
But notice that, because $G_+(x)$ is Lorentz invariant, it must be an even function, $G_+(x) = G_+(-x)$, for spacelike x.

There is a (proper) Lorentz transformation that takes $x^0 \to -x^0$ and $\vec{x} \to -\vec{x}$. (Note that G_+ does not have this property for x timelike. It is not invariant under time reversal because only *positive* k^0 appears). Hence

$$[\phi(x), \phi(y)] = 0 \qquad (x-y)^2 < 0$$

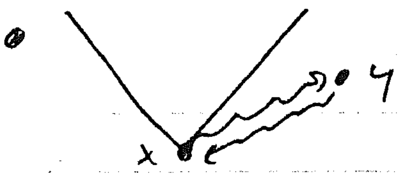
The fields *are* causal observables.

Causality actually results from a remarkable interference effect. There *is* (in a sense) propagation outside the light cone from x to y, but there is also propagation from y to x.



Because the amplitudes for these processes interfere *destructively*, measurements at x and y do not influence one another.

Remarks

- 

  If we had considered a theory of particles that carry the value Q of a conserved charge, then propagation of a particle with charge $-Q$ would be necessary to destructively interfere with propagation of a charge-Q particle outside the light cone. Together, causality and the positivity of the energy require, in a relativistic theory, the existence of antiparticles.

- Causality severely restricts the algebra of observables of a relativistic theory. With the phase convention that we adopted, we found

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2k^0}\left[e^{-ik\cdot x}A(k) + e^{ik\cdot x}A(k)^\dagger\right].$$

But had we adopted a different convention, we could have had

$$\phi^\theta(x) = \int \frac{d^3k}{(2\pi)^3(2k^0)}\left[e^{-ik\cdot x}e^{-i\theta}A(k) + e^{ik\cdot x}e^{i\theta}A(k)^\dagger\right].$$

Including both $\phi(x)$ and $\phi^\theta(x)$ as local observables of the theory is worse than redundant, for ϕ and ϕ^θ are not local relative to one another unless $e^{-i\theta} = \pm 1$

We have $[\phi^\theta(x), \phi(y)] = e^{-i\theta}\Delta_+(x-y) - e^{i\theta}\Delta_+(y-x)$

$$= -2i\sin\theta\, \Delta_+(x-y), \text{ for } (x-y)^2 < 0$$

So we see that the algebra of observables is an exclusive club — No operator may join unless it commutes, at spacelike separation, with all operators that already belong. In particular, ϕ^θ $(e^{-i\theta} \neq \pm 1)$ cannot be admitted if ϕ is already a member.

Some further properties of the field theory —

<u>Field equation</u>

observe that $\phi(x)$ obeys a covariant wave equation

$(\partial^\mu\partial_\mu + m^2)\phi(x) = 0$ — the <u>Klein-Gordon equation</u>

This wave eqn is a transcription of the mass shell condition

$$p^\mu p_\mu = m^2$$

satisfied by the particles.

The eqn has positive frequency solns $e^{-ik\cdot x}$
and negative frequency solns $e^{ik\cdot x}$
$(k^2 = m^2)$

The negative frequency solutions caused confusion when $\phi(x)$ was interpreted as a single-particle <u>wave function</u>. But we have seen that they have a simple and natural interpretation if ϕ is an operator that creates and destroys particles.

4-momentum operator

The 4-momentum P^μ satisfying $P^\mu|k\rangle = k^\mu|k\rangle$ can be expressed in terms of A and A^+

It is $$P^\mu = \int \frac{d^3k}{(2\pi)^3 2k^0}\, k^\mu A(k)^+ A(k)$$

The properties $P^\mu|0\rangle = 0$

and $[P^\mu, A(k)] = -k^\mu A(k)$

(which follows from $e^{iP\cdot a}A(k)e^{-iP\cdot a} = e^{-ik\cdot a}A(k)$) are easily verified

Conventional Normalization

Although it is obviously convenient to expand

$$\phi = \int \frac{d^3k}{(2\pi)^3 2k^0}\left[e^{-ik\cdot x}A(k) + e^{ik\cdot x}A(k)^+\right]$$

in terms of creation and annihilation operators for relativistically normalized states, it is more conventional to introduce

$$a(k) = \frac{A(k)}{(2\pi)^{3/2}(2k^0)^{\frac{1}{2}}},$$

so that

$$[a(k), a(k')^+] = \delta^3(\vec{k}-\vec{k}')$$

Then we have

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}(2k^0)^{\frac{1}{2}}} \left[e^{-ik\cdot x} a(k) + e^{ik\cdot x} a(k)^\dagger \right]$$

Alternate Construction of Hilbert space

The construction of the one-particle Hilbert space $\mathcal{H}^{(1)}$ can be described in an alternative language that, while superfluous now, will prove useful when we consider the case of field theory on a curved background.

Note first that, once a local field is constructed, the overlap of an arbitrary state with $\phi(x)|0\rangle$, for all x, provides a natural 1-1 map

$$\mathcal{H}^{(1)} \simeq \{\underline{\text{positive}} \text{ frequency solutions to the Klein-Gordon eqn}\}$$

The state

$$|\tilde{f}\rangle = \int \frac{d^3k}{(2\pi)^3 2k^0} \tilde{f}(k) |k\rangle$$

is associated with the solution

$$\langle 0|\phi(x)|k\rangle = \int \frac{d^3k}{(2\pi)^3 2k^0} \tilde{f}(k) e^{-ik\cdot x}$$

Equivalently, $\langle 0|\phi(x)|k\rangle^* = \langle k|\phi(x)|0\rangle$ provides the 1-1 map

$$\mathcal{H}^{(1)} \simeq \{\underline{\text{negative}} \text{ frequency solns to K.G. eqn}\}$$

This observation can also serve as the starting point of the construction of the scalar quantum field theory. The classical field theory is defined by the classical Klein-Gordon field equation

$$(\partial^\mu \partial_\mu + m^2)\, \phi(x) = 0$$

To obtain a quantum theory, we begin by constructing the "one-particle" Hilbert space $\mathcal{H}^{(1)}$.

The general solution to the K-G. eqn can be expanded in the basis

$$u_k(x) = e^{-ik\cdot x} \quad \text{- positive frequency}$$
$$u_k(x)^* = e^{ik\cdot x} \quad \text{- negative frequency}$$

where $\qquad k^2 = m^2, \quad k^0 \geq 0.$

If Lorentz transformations act on the solutions according to

$$\Lambda: \ f(x) \to f(\Lambda^{-1}x)$$

(Note: the opposite of how the quantum field ϕ transforms), then the positive frequency and negative frequency basis each transform irreducibly under the proper Lorentz transformations:

$$\Lambda: \ u_k(x) \to u_k(\Lambda^{-1}x) = u_{\Lambda k}(x)$$
$$\Lambda: \ u_k(x)^* \to u_k(\Lambda^{-1}x)^* = u_{\Lambda k}(x)^*$$

The proper Lorentz transformations do not

mix up positive and negative frequency.
So the positive (or negative) frequency solutions are a linear space on which Lorentz transformations act irreducibly. (And translations, too.)

Solutions of the KG eqn are also in 1-1 correspondence with initial-value data: the solution to $(\partial^\mu \partial_\mu + m^2) f(x) = 0$ is uniquely determined by values of f and $\dot{f}$ on a surface $x^0 = t =$ constant. But for a solution of definite frequency (e.g. positive), the $\dot{f}$ initial data is not necessary to propagate f away from the initial-value surface. There are 1-1 correspondences

$$\{\text{pos. freq. solns.}\} \simeq \{\text{neg. freq. solns.}\} \simeq \{\text{functions on } \mathbb{R}^3\}.$$

To obtain a Hilbert space, we must specify an inner product. To define the inner product of two solutions, specify a time t, and integrate over the time slice

$$(f, g) = i \int_t d^3x \left[f^*(x)\, \partial_t g(x) - \partial_t f(x)^*\, g(x) \right]$$

(Note: Birrell and Davies define $(f,g)^* =$ above.)
Then, for the basis $u_k(x) = e^{-ik\cdot x}$, we have

$$(u_k, u_{k'}) = (2\pi)^3\, 2k^0\, \delta^3(\vec{k} - \vec{k}')$$
$$(u_k, u_{k'}^*) = 0$$
$$(u_k^*, u_{k'}^*) = -(2\pi)^3\, 2k^0\, \delta^3(\vec{k} - \vec{k}')\,.$$

This Klein-Gordon inner product is not positive definite on the space of all solutions. But it is positive definite on the space of pos. freq. solns. Furthermore, $-(f,g)$ is pos. def. for neg. freq. solns, and the pos. and neg. frequency solutions are orthogonal to each other. There is a natural decomposition of the space of all solutions

$$\{\text{solutions}\} = \{\text{pos. freq.}\} \oplus \{\text{neg. freq}\},$$

such that the direct sum is a sum of spaces that are <u>orthogonal</u> with respect to the K. G. inner product.

The basis $u_K(x)$ for the positive frequency solns has precisely the normalization of the plane wave basis for $\mathcal{H}^{(1)}$

$$\langle k | k' \rangle = (2\pi)^3 (2k^0)\, \delta^3(\vec{k}-\vec{k}'),$$

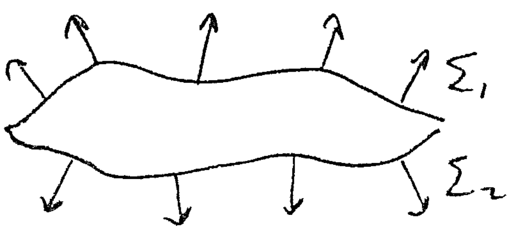
so the $u_k(x)$'s correspond to relativistically normalized one-particle planewave states

In defining (f,g), we seem to have picked out a particular frame and a particular slice. But we can rewrite (f,g) in a form that is manifestly invariant under deformations of the spacelike surface.

we write

$$(f,g) = i\int_{\Sigma} d^3x\, n^\mu [f^* \partial_\mu g - \partial_\mu f^* g]$$

– integrated over a spacelike 3-dimensional surface, where n^μ is the normalized unit (timelike) normal to Σ (in the forward light cone).



If we distort the surface Σ_1 to a new surface Σ_2, then

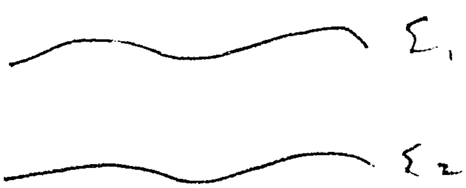
$$(f,g)_{\Sigma_1} - (f,g)_{\Sigma_2} = i\int_{\Sigma_1-\Sigma_2} d^3x\, n^\mu (f^* \partial_\mu g - \partial_\mu f^* g)$$

$$= i\int_{\Omega} d^4x\, \partial^\mu (f^* \partial_\mu g - \partial_\mu f^* g), \text{ where } \partial\Omega = \Sigma_1 - \Sigma_2$$

(divergence theorem). But since f and g are solutions to the KG eqn,

$$\partial^\mu (f^* \partial_\mu g - \partial_\mu f^* g) = f^* \partial^2 g - \partial^2 f^* g$$
$$= (m^2 - m^2) f^* g = 0$$

– so $(f,g)_{\Sigma_1} - (f,g)_{\Sigma_2} = 0$



The conclusion holds if $\Sigma_1 - \Sigma_2$ is not a closed surface, if Σ_1 and Σ_2 are infinite spacelike planes and there is no contribution to the integral from spatial infinity (true for normalized states).

We have now described how, beginning with the classical field eqn, we may construct $\mathcal{H}^{(1)}$ as a space of positive frequency solutions. Once we have $\mathcal{H}^{(1)}$, we can obtain $\mathcal{H}^{(n)}$ as a symmetrized tensor product — e.g.,

$$\mathcal{H}^{(2)} = \mathcal{H}^{(1)} \otimes_s \mathcal{H}^{(1)}$$

Then we proceed to construct $A(k)$ and $\phi(x)$ as before, obtaining

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2k^0} \left[u_k(x) A(k) + u_k(x)^* A(k)^\dagger \right].$$

Relation to Canonical Method

Recall that

$$[\phi(x), \phi(y)] = G_+(x-y) - G_+(y-x)$$
$$= \int \frac{d^3k}{(2\pi)^3 2k^0} \left[e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right]$$

And therefore

$$[\phi(x), \dot\phi(y)] = \int \frac{d^3k}{(2\pi)^3 2k^0} \left[ik^0 e^{-ik\cdot(x-y)} + ik^0 e^{ik\cdot(x-y)} \right]$$

$$[\dot\phi(x), \dot\phi(y)] = \int \frac{d^3k}{(2\pi)^3 2k^0} (k^0)^2 \left[e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right]$$

If we evaluate these at equal times $x^0 - y^0 = 0$, we find

$$[\phi(x), \phi(y)]_{et} = 0$$

$$[\phi(x), \dot{\phi}(y)]_{et} = i\delta^3(\vec{x}-\vec{y})$$

$$[\dot{\phi}(x), \dot{\phi}(y)]_{et} = 0$$

Since $G_+(x)$ is a Lorentz-invariant function, these identities hold in all (inertial) reference frames. They recall the commutation relations

$$[q_i, q_j] = 0, \quad [q_i, p_j] = i\delta_{ij}, \quad [p_i, p_j] = 0,$$

of a canonical quantum mechanical system ($\hbar = 1$), except with continuum normalization. (The continuum normalization reminds us that the fields are <u>distributions</u>. We could obtain the discrete normalization by smearing the fields with some complete set of square integrable functions.)

These canonical commutation relations are equivalent to the commutation relations satisfied by the $A(k)$'s and $A(k)^+$'s. If we Fourier transform, we have

$$\tilde{\phi}(t, \vec{k}) = \int d^3x \; e^{-i\vec{k}\cdot\vec{x}} \phi(t, \vec{x})$$

$$\tilde{\phi}(t, \vec{k})^+ = \tilde{\phi}(t, -\vec{k})$$

and $$[\tilde{\phi}, \tilde{\phi}]_{et} = [\dot{\tilde{\phi}}, \dot{\tilde{\phi}}]_{et} = 0$$

$$[\tilde{\phi}(t, \vec{k}), \dot{\tilde{\phi}}(t, -\vec{k}')] = i(2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

where $$\tilde\phi(t,\vec{k}) = \frac{1}{2k^0}\left[e^{-ik^0t}A(k^0,\vec{k}) + e^{ik^0t}A(k_0,-\vec{k})^\dagger\right].$$

$$\dot{\tilde\phi}(t,\vec{k}) = \frac{-i}{2}\left[e^{-ik^0t}A(k^0,\vec{k}) - e^{ik^0t}A(k_0,-\vec{k})^\dagger\right]$$

and hence

$$e^{-ik^0t}A(k) = k^0\tilde\phi(t,\vec{k}) + i\,\dot{\tilde\phi}(t,\vec{k}),$$

adjoint $\Rightarrow$ $$e^{ik^0t}A(k)^\dagger = k^0\tilde\phi(t,-\vec{k}) - i\,\dot{\tilde\phi}(t,-\vec{k})$$

So the commutation relations

$$[A(k), A(k')] = 0 = [A(k)^\dagger, A(k')^\dagger].$$

$$[A(k), A(k')^\dagger] = (2\pi)^3 2k^0\,\delta^3(\vec{k}-\vec{k}')$$

can evidently be recovered from the equal time commutators of ϕ and $\dot\phi$ (as well as vice-versa).

To complete the specification of the canonical system, we need a Hamiltonian H expressed in terms of ϕ and its conjugate momentum $\pi = \dot\phi$. H is P^0, the generator of time evolutions satisfying

$$[P^0, A(k)] = -k^0A(k)$$

$$[P^0, A(k)^\dagger] = k^0A(k)^\dagger$$

and therefore $$[H, \phi(x)] = -i\,\dot\phi(x)$$

$$[H, \pi(x)] = -i\,\dot\pi(x)$$

We have $$H = \int \frac{d^3K}{(2\pi)^3 2K^0} K^0 A(K)^\dagger A(K)$$

where

$$A(K)^\dagger A(K) = \tilde{\pi}(t,\vec{K})\tilde{\pi}(t,-\vec{K}) + (K^0)^2 \tilde{\phi}(t,\vec{K})\tilde{\phi}(t,-\vec{K})$$
$$+ iK^0[\tilde{\phi}(t,-\vec{K})\tilde{\pi}(t,\vec{K}) - \tilde{\pi}(t,-\vec{K})\tilde{\phi}(t,\vec{K})]$$

So $$H = \int \frac{d^3K}{(2\pi)^3} \frac{1}{2}\left[\tilde{\pi}(t,\vec{K})\tilde{\pi}(t,-\vec{K}) + (K^0)^2\tilde{\phi}(t,\vec{K})\tilde{\phi}(t,-\vec{K}) - K^0(2\pi)^3\delta^3(0)\right]$$

Up to an additive constant, this is the Hamiltonian of an infinite set of uncoupled harmonic oscillators, where the oscillator labeled $\vec{K}$ has frequency $\omega_K = K^0 = \sqrt{\vec{K}^2 + m^2}$.

To understand the origin of the constant, note that $(2\pi)^3\delta^3(0) = \int d^3x\,(1) = V$ is the spatial volume, and $d^3K/(2\pi)^3$ is the density of oscillator modes per unit volume; so

$$H_0 = \text{constant} = -\sum_{\text{modes}} \left(\tfrac{1}{2}\omega_K\right)$$

— the constant subtracts away the zero point energy of all the oscillators. We make this subtraction so that $P^\mu|0\rangle = 0$.

Otherwise, we would have to split P^μ up into two pieces, where one piece transforms as a four vector, and the remainder, $\langle 0 | P^\mu | 0 \rangle$, is invariant under Lorentz transformations.

Canonical Quantization

The canonically quantized theory can be arrived at starting from a classical theory defined by an action principle.

To define a relativistic classical field theory, we may specify that $\phi(x)$ is a scalar field

$$(\Lambda, a) : \phi(x) \to \phi(\Lambda x + a)$$

and construct

$$S = \int d^4x \, \mathcal{L}(\partial_\mu \phi(x), \phi(x))$$

We require

- S is local and a functional of ϕ and first derivatives, so that initial-value problem is well formulated.
- $\mathcal{L}$ is Poincaré invariant (frame independent dynamics)
- $\mathcal{L}$ is quadratic in ϕ and $\partial^\mu \phi$ (linear equation of motion; free field theory)

The Lagrange density is --

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2$$

This is unique (for $m^2 \neq 0$) — up to linear redefinition of ϕ. We remove any term linear in ϕ by $\phi \to \phi + b$, and remove coef of the first term by $\phi \to c\phi$. Positivity of energy will require $m^2 \geq 0$ and positive coef of $\tfrac{1}{2}\partial^\mu\phi\,\partial_\mu\phi$ term. For $m^2 = 0$, no linear term is allowed.

We obtain field equation from action principle

$$0 = \delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\,\delta\partial_\mu\phi\right]$$

$$= \int d^4x\ \delta\phi\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right] + \text{surface term}$$

$$\Rightarrow \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = 0$$

if $\delta S = 0$ for arbitrary variations that vanish at the boundary.

For above $\mathcal{L}$, we have $(\partial_\mu\partial^\mu + m^2)\phi(x) = 0$ — the Klein-Gordon equation.

We construct the canonical Hamiltonian

$$H \quad \sum_i \dot{q}_i p_i - L, \qquad p_i = \frac{\partial L}{\partial\dot{q}_i}$$

$$\Rightarrow H \quad \int d^3x\,(\dot{\phi}\pi - \mathcal{L}), \qquad \pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$

— where $\dot{\phi}$ is eliminated in favor of π

For the free scalar field theory we have

$$\pi(x) = \dot{\phi}(x) \text{ and}$$

$$H(\phi, \pi) = \int d^3x \left[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\vec{\nabla}\phi)^2 + \tfrac{1}{2}m^2\phi^2 \right]$$

We obtain a quantum field theory by requiring that ϕ and π obey canonical commutation relations at equal times (they are to be regarded as Heisenberg operators)

$$[\phi, \phi]_{et} = [\pi, \pi]_{et} = 0$$

$$[\phi(x), \pi(y)]_{et} = i\delta^3(\vec{x} - \vec{y})$$

The $\phi(t, \vec{x})$ at fixed t may be regarded as a complete set of commuting observables for the canonical system.

If we expand ϕ and $\dot{\phi} = \pi$ in terms of A and A^+, we obtain again the quantum theory described previously (e.g. p 1.32-33). We chose a particular frame in which to canonically quantize, but the commutation relations are the same in all inertial frames, which is enough to ensure that the theory is causal (ϕ's commute at spacelike separation.)

Since the theory is Poincare invariant, we can construct conserved quantities by the Noether procedure. E.g., from translation invariance we obtain

$\partial_\nu T^{\mu\nu} = 0$, where $T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$ is the canonical energy momentum tensor. We will not derive this here, as we will soon derive $T^{\mu\nu}$ by another method.

The conserved four-momenta are then

$$P^\mu = \int d^3x \, T^{\mu 0}$$

There is an ordering ambiguity in $P^0 = H$, which, as we noted before, can be resolved by demanding $\langle 0 | P^\mu | 0 \rangle = 0$. The corresponding ordering prescription is called normal ordering.

- Note: canonical quantization is a noncovariant procedure. We must choose a frame to define t and a Hamiltonian H. But we have seen that the theory that we obtain is covariant (admits a unitary representation of Poincare)

- The canonical quantization procedure gives a quantum theory that agrees with a given classical theory in the classical limit. (Hamilton equations are satisfied at the operator level:

$$\dot{q} = -i[q, H] = \frac{\partial H}{\partial p} \qquad \left(q \sim i \frac{\partial}{\partial p}\right)$$
$$\dot{p} = -i[p, H] = -\partial H / \partial q \qquad)$$

Thus, it is natural to apply this procedure to obtain a quantum version of a field that is observable classically – e.g. the electromagnetic field. Canonical quantization is much less natural if we are trying to devise a relativistic theory of pions or electrons.

- Canonical quantization has an important advantage over our other procedures – it is easily applied to an interacting (nonlinear) theory. Our construction of the Klein-Gordon inner product required that the field equation be linear (in order that it be slice independent). And our Fock space construction assumed that the particles are noninteracting.