3. Quantum Field Theory in Rindler Spacetime

Our next topic is "How does the Minkowski space vacuum look to a uniformly accelerated observer?" After all the fuss about curved spacetime, it is disappointing to return to flat spacetime, but there is ample motivation.

- quantum field fluctuations that have positive frequency according to a clock of an inertial observer can have negative frequency as measured by a noninertial observer. The uniformly accelerated observer provides us with a nontrivial and calculable example of a Bogoliubov transformation.

- The principle of equivalence asserts that physics as seen by a uniformly accelerated observer is the same as physics in a uniform magnetic field, so we will be investigating a gravitational effect on quantum fields

- This example provides a first glimpse of a profound connection among gravitation, quantum mechanics, and thermodynamics (for the accelerated observer sees a thermal radiation bath) An even deeper connection of this kind will emerge when we study black holes.

- invoking analogies with the simpler Rindler spacetime will guide our understanding of more subtle spacetimes (e.g. Schwarzschild, de Sitter).

Before considering the effect of quantum field fluctuations on an accelerated observer, let's first consider some features of field fluctuations in an inertial frame. We'll continue to consider a theory of a real free scalar field, and to keep things as simple as possible, we'll suppose that the field is massless ($m^2=0$)

In a homework problem, you evaluated

$$G_+(x) = \int \frac{d^3k}{(2\pi)^3 2k^0} e^{-ik\cdot x} = \langle 0|\,\phi(x)\,\phi(0)\,|0\rangle$$

where $m^2=0 \Rightarrow k^0 = |\vec{k}|$, and found

$$G_+(x) = \frac{1}{8\pi^2 r}\left(\frac{1}{r-(t-i\epsilon)} + \frac{1}{r+(t-i\epsilon)}\right)$$

(where $t = x^0$, $r = |\vec{x}|$)

$$= \frac{-1}{4\pi^2(x^2 - 2i\epsilon x^0)}$$

Here we have invoked the observation that $D_+(x)$ is analytic for $\mathrm{Im}\, t<0$, and so have defined the integral by giving t a small negative imaginary part $-i\epsilon$.

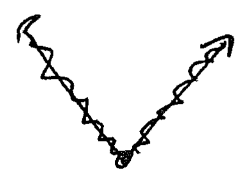
using $\frac{1}{a+i\epsilon} - \frac{1}{a-i\epsilon} = -2\pi i\,\delta(a)$,

you can also evaluate

$$iG(x) = [\phi(x), \phi(0)] = G_+(x) - G_+(-x)$$
$$= \frac{-i}{4\pi r}\left[\delta(r-t) - \delta(r+t)\right].$$

The field commutator, in the $m^2 = 0$ case, has support only on the light cone $t = \pm r$. This makes sense, as massless particles propagate information at only the speed $c = 1$.

Our understanding of the form of the field commutation relations can be made more quantitative. We may attempt to measure (smeared) field in vicinity of x by observing the response of a test charge that couples to $\phi(x)$. (This actually measures $\vec{\nabla}\phi(x)$.) We obtain a precise measurement of the field by a precise measurement of the impulse received by the charge, but because we measure momentum, the position of the charge undergoes uncontrollable quantum fluctuations. Hence, the charge accelerates, and an accelerated charge radiates. The field commutator is just a difference of retarded and advanced Green functions, describing the $1/r$ decay of the ϕ wave front propagating along the lightcone, which interferes with a subsequent measurement of $\phi(x)$ there.

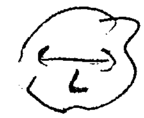


the function $G_+(x) = \langle 0|\phi(x)\phi(0)|0\rangle$

$$= \frac{-1}{4\pi^2(x^2 - i\epsilon x^0)}$$

blows up as $x^2 \to 0$, as does the expectation value of $\phi(x)\phi(0)$ in any Fock space state. This divergence cautions us again that $\phi(x)$ is not itself an observable; observables are obtained by smearing fields — integrating them against smooth functions. $\phi(x)$ cannot be measured because, although its mean value is zero, its dispersion is ∞. (The product of two distributions need not be a well-defined distribution.)

If we measure the field smeared with a test function f_L whose support has a linear size of order L, then



$$\langle 0|\,\phi(f_L)^2|0\rangle \sim \frac{1}{L^2}$$ so typical value of $\phi(f_L)$ is $\sim \frac{1}{L}$

Similarly, typical value of $\nabla\phi(f_L) \sim \frac{1}{L^2}$, and of energy in region with spatial volume L^3: $E_L = \int d^3x\, \frac{1}{2}(\nabla\phi)^2 \sim \frac{1}{L}$.

These fluctuations of the fields in the vacuum are not surprising — the position of a harmonic oscillator also oscillates in the ground state (zero point oscillation), and the (free) field is just a set of uncoupled oscillators.

For $\quad x(t) = \frac{1}{\sqrt{2\omega}}\left(e^{-i\omega t} a + e^{i\omega t} a^\dagger\right),$

we have $\quad \langle 0 | x(t)\, x(0) | 0 \rangle = \frac{1}{2\omega} e^{-i\omega t}.$

Similarly, if we Fourier analyze the field,

$$\tilde\phi(t,\vec k) = \int d^3x\, e^{-ik\cdot x}\phi(t,\vec x)$$
$$= \frac{1}{2k^0}\left[e^{-ik^0 t} A(k^0,\vec k) + e^{ik^0 t} A(k^0,\vec k)\right]$$

(page 1.32), then

$$\langle 0 | \tilde\phi(t,\vec k)\, \tilde\phi(0,\vec k') | 0 \rangle = \frac{1}{2k^0} e^{-ik^0 t} (2\pi)^3 \delta^3(\vec k - \vec k')$$

Because of the translation invariance, the fluctuations are diagonal in $\vec k$; each mode of the field fluctuates independently (no correlations).

Fluctuations in a region of size $\sim 2\pi/k$ are dominated by wave number $\sim k$.

$$[\Delta\phi(\vec k)]^2 = \frac{k^3}{(2\pi)^3}\int d^3x\, e^{-i\vec k\cdot\vec x}\langle 0 | \phi(t,\vec x)\, \phi(t,0) | 0 \rangle$$

fluctuation at fixed time

$$= \frac{k^3}{(2\pi)^3}\,\frac{1}{2k^0} \sim \frac{1}{L^2} \quad \left(L = \frac{2\pi}{k}\right).$$

The field admits a "classical" description only if the fluctuations in ϕ are small compared to the mean value of ϕ. Classical field and field quanta are "complementary" concepts. Measurement of the field with high spatial resolution typically produces many quanta. And for the spatially averaged field to behave classically, the number of field quanta per (wavelength)^3 must be large. (Fluctuations cause $(\Delta E)_L \sim \frac{1}{L}$, and $E(\text{wavelength } L) \sim \frac{1}{L}$, so $\Delta E/E \sim \frac{1}{n}$ if there are n quanta.)

To describe measurements in more detail, consider a "particle detector" that can absorb and emit field quanta.

Idealize the detector as pointlike, and suppose it travels along a worldline $x^\mu(\tau)$, parametrized by τ, the proper time along its world line. The Hamiltonian of the coupled detector-field system is

$$H = (H_0)_{\text{detector}} + (H_0)_{\text{field}} + \lambda\, \mathcal{M}\, \phi(x(\tau))$$

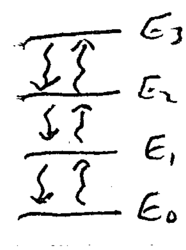
this is H in the instantaneous reference frame of the detector

this Hamiltonian acts on a Hilbert space

$$\mathcal{H} = \mathcal{H}_{\text{detector}} \otimes \mathcal{H}_{\text{field}}.$$

$\mathcal{M}$ is a perturbation of $(H_0)_{\text{detector}}$ that is capable of exciting a transition in the detector, and λ is a coupling constant, assumed small so that we can use perturbation theory.

The unperturbed detector has various energy eigenstates, and the coupling of the detector to the field allows



the detector to become excited by absorbing a field quantum, or to de-excite by emitting a quantum. To lowest order in λ, the amplitude for a transition is

$$A = \langle \text{final} | -i \int_{-\infty}^{\infty} d\tau \, \lambda M_I(\tau) \, \phi(x(\tau)) \, | \text{initial} \rangle$$

where $$M_I(\tau) = e^{i(H_0)_{det}\tau} M e^{-i(H_0)_{det}\tau}$$

is the interaction picture detector perturbation (τ is the detector's proper time that appears here, as M is the perturbation of H_{det} in the rest frame of the detector.)

If the initial state of the detector is an energy eigenstate with energy E, and and its final state is an energy eigenstate with energy $E+\omega$, we have

$$A = (-i\lambda) \langle E+\omega | M | E \rangle \int_{-\infty}^{\infty} d\tau \, e^{i\omega\tau} \langle \text{final field} | \phi(x(\tau)) | \text{initial field} \rangle$$

Now suppose we observe only the change in the state of the detector. The probability of this change is given by squaring the amplitude, and summing over the final states of the field

$$\text{Prob}(E \to E+\omega) = \lambda^2 |\langle E+\omega | M | E \rangle|^2$$

$$\int_{-\infty}^{\infty} d\tau \, e^{i\omega\tau} \int_{-\infty}^{\infty} d\tau' \, e^{-i\omega\tau'} \sum_f \langle i | \phi(x(\tau')) | f \rangle \langle f | \phi(x(\tau)) | i \rangle$$

$$\text{Prob}(E \to E+\omega) = \lambda^2 |\langle M\rangle|^2 \int d\tau\, d\tau'\, e^{-i\omega(\tau'-\tau)} \langle i|\phi(x(\tau'))\,\phi(x(\tau))|i\rangle$$

If $|i\rangle$ is a momentum eigenstate, $U(a)|i\rangle = e^{iP_i \cdot a}|i\rangle$, then

$$\text{Prob}(E \to E+\omega) = \lambda^2 |\langle M\rangle|^2 \int d\tau\, d\tau'\, e^{-i\omega(\tau'-\tau)} \langle i|\phi[x(\tau')-x(\tau)]\,\phi(0)|i\rangle$$

Suppose now that the detector moves inertially. Then $x^\mu(\tau') - x^\mu(\tau) = u^\mu(\tau'-\tau)$ where u^μ is the 4-velocity. Then one τ integral is trivial, and gives a factor of elapsed proper time. So

$$\text{Rate} = \frac{\text{Prob}(E \to E+\omega)}{\text{Proper time}} = \lambda^2 |\langle M\rangle|^2 \int d\tau\, e^{-i\omega\tau} \langle i|\phi(x(\tau))\phi(0)|i\rangle$$

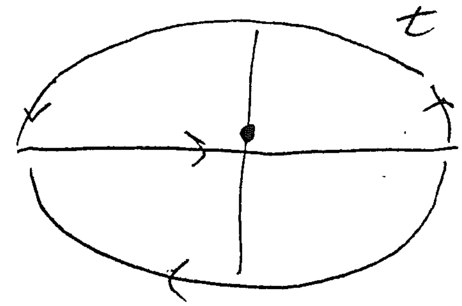
And if the field is initially in its vacuum state, we have

$$\text{Rate} = \lambda^2 |\langle M\rangle|^2 \int d\tau\, e^{-i\omega\tau}\, G_+(x(\tau))$$

This is most easily evaluated in the rest frame of the detector. (The only effect of boosting to another frame is to red shift the emitted radiation.) Recalling

$$G_+(x) = \frac{-1}{4\pi^2\left[(t-i\epsilon)^2 - \vec{x}^{\,2}\right]},$$

we readily evaluate the Fourier transform



by contour integration. For $\omega > 0$, we complete contour in LHP, and for $\omega < 0$, we complete in UHP (enclosing the pole):

$$\int dt\, e^{-i\omega t} \frac{-1}{4\pi^2 (t-i\epsilon)^2} = \begin{cases} 0\,, & \omega > 0 \\ -\frac{1}{4\pi^2}(2\pi i)\frac{d}{dt}e^{-i\omega t}\Big|_{t=0} = -\frac{\omega}{2\pi}\,, & \omega < 0 \end{cases}$$

$$= \theta(-\omega)\,\frac{-\omega}{2\pi} = \theta(-\omega)\,\frac{|\omega|}{2\pi}$$

we find then,

$$\text{Rate} = 0 \quad \text{for } \omega > 0$$

– Naturally, the inertial detector sees no quanta when moving in the vacuum

$$\text{Rate} = d^2 |\langle M \rangle|^2 \frac{|\omega|}{2\pi} \quad \text{for } \omega < 0$$

– the detector can deexcite by emitting a quantum. This rate is just that given by Fermi's rule

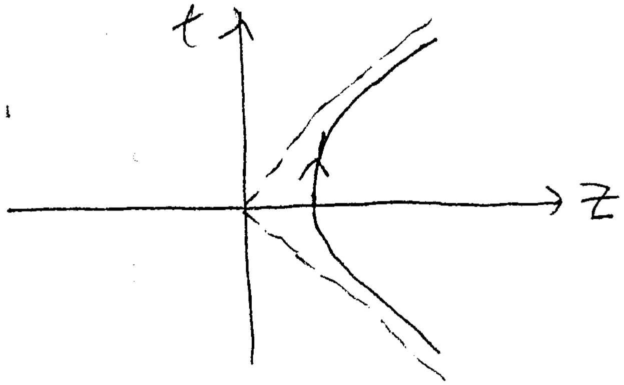
$$\text{Rate} = 2\pi\, |H'|^2 \times \left(\begin{array}{c}\text{density of final states}\\ \text{per unit energy}\end{array}\right),$$

since the relativistic density of states is just

$$\frac{d^3K}{2K^0(2\pi)^3} = \frac{4\pi K^2 dK}{2K^0(2\pi)^3} = \frac{\omega\, d\omega}{4\pi^2}$$

(for $m^2 = 0 \Rightarrow \omega = |\vec{K}|$)

We turn now to the case of a <u>uniformly accelerated</u> detector, i.e. a detector moving with constant proper acceleration as measured in its instantaneous inertial frame.



An object accelerating uniformly in the $+\hat{z}$ direction moves on a world line

$$z^2 - t^2 = \frac{1}{a^2}$$

— a hyperbola in the z-t plane, where a is the proper acceleration. (Motion is hence sometimes called "hyperbolic")

Parametrized by proper time τ, the world line is

$$z(\tau) = \frac{1}{a}\cosh(a\tau)$$
$$t(\tau) = \frac{1}{a}\sinh(a\tau)$$

From
$$dz = (\sinh a\tau)\, d\tau$$
$$dt = (\cosh a\tau)\, d\tau$$

we verify $\quad d\tau^2 = dt^2 - dz^2 \quad$ — τ is indeed proper time

the instantaneous velocity is

$$v(\tau) = \frac{dz}{dt} = \tanh(a\tau)$$

Recall that relativistic velocities add like tanh's:

$$\left.\begin{array}{l} v_1 = \tanh\theta_1 \\ v_2 = \tanh\theta_2 \end{array}\right\} \Rightarrow v = \tanh(\theta_1 + \theta_2)$$

So as $\tau \to \tau + d\tau$, the velocity as measured in frame of the moving object, changes by

$$dv = \tanh(a\,d\tau) = a\,d\tau$$

This shows that a is the proper acceleration

(proper acceleration is same in all inertial frames.)

Note that the world line, the hyperbola, is preserved by a Lorentz boost along $\hat{z}$. In fact, the effect of a boost is

$$\begin{pmatrix} t \\ z \end{pmatrix} \to \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} \frac{1}{a}\sinh a\tau \\ \frac{1}{a}\cosh a\tau \end{pmatrix} = \begin{pmatrix} \frac{1}{a}\sinh(a\tau+\theta) \\ \frac{1}{a}\cosh(a\tau+\theta) \end{pmatrix}$$

— this is equivalent to a shift in τ by a constant. In other words, the tangent to the world line is just a Lorentz boost generator:

$$\frac{1}{a}\frac{\partial}{\partial\tau} = \text{boost generator}$$

Return now to the formula on page 3.8

$$\text{Prob}(E \to E+\omega) = \lambda^2 |\langle M\rangle|^2 \int d\tau\, d\tau'\, e^{-i\omega(\tau'-\tau)} \langle 0|\phi(x(\tau'))\,\phi(x(\tau))|0\rangle$$

(where field is initially in the vacuum state)

For a uniformly accelerated observer

$$x^\mu(\tau) = \Lambda(\theta = a\tau)\, x^\mu(\tau=0)$$

$\uparrow$ boost along $\hat{z}$

Since ϕ is a scalar field, $\phi(\Lambda x) = U(\Lambda)\phi(x)U(\Lambda^{-1})$ and the vacuum is Lorentz invariant, $U(\Lambda)|0\rangle = |0\rangle$, we have

$$\langle 0|\phi(x(\tau'))\,\phi(x(\tau))|0\rangle$$
$$= \langle 0|\phi(x(0))\,U(\Lambda(\theta'))^{-1}\,U(\Lambda(\theta))\,\phi(x(0))|0\rangle$$

and $$\Lambda(\theta')^{-1}\Lambda(\theta) = \Lambda(\theta-\theta')$$

so $$= \langle 0|\phi(x(0))\,U(\Lambda(\theta-\theta'))\,\phi(x(0))|0\rangle$$

Therefore, as for the case of an inertial observer, this matrix element is a function of $\tau'-\tau$, and we obtain a constant rate per unit proper time

$$\text{Rate} = \lambda^2|\langle M\rangle|^2\int d\tau\, e^{-i\omega\tau}\langle 0|\phi(x(\tau))\,\phi(x(0))|0\rangle$$

$$= \lambda^2|\langle M\rangle|^2\int d\tau\, e^{-i\omega\tau}\, G_+(x(\tau)-x(0))$$

But now we evaluate

$$G_+(x) = \frac{-1}{4\pi^2[(t-i\epsilon)^2-\vec{x}^2]}$$

along the hyperbola

$$x(\tau)-x(0) = \left[\tfrac{1}{a}\sinh(a\tau),\ \tfrac{1}{a}\cosh(a\tau)-1\right]$$

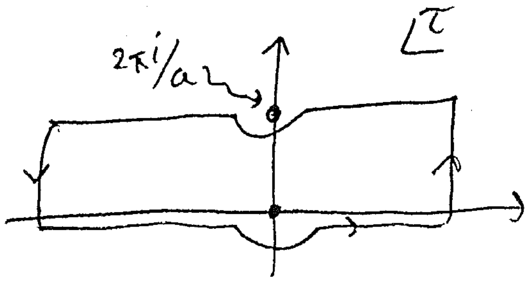
and so $(t-i\epsilon)^2 - z^2$

$$= \frac{1}{a^2}\left[\sinh^2(a\tau) - \cosh^2(a\tau) - 1 + 2\cosh(a\tau) - i\epsilon\sinh a\tau\right]$$

$$= \frac{4}{a^2}\sinh^2\left(\frac{a\tau}{2}\right) - i\epsilon\sinh(a\tau) \simeq \frac{4}{a^2}\sinh^2\left(\frac{a\tau}{2}-i\epsilon\right)$$

or

$$\text{Rate} = \lambda^2 |\langle M \rangle|^2 \int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau} \left[\frac{-a^2}{16\pi^2 \sinh^2(\frac{a\tau}{2} - i\epsilon)} \right]$$



To evaluate

$$\Pi(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau} \frac{-a^2}{16\pi^2 \sinh^2(\frac{a\tau}{2} - i\epsilon)}$$

consider the contour shown. Since $\sinh^2(x)$ is periodic with period $i\pi$, and the $\sinh^{-2}$ prevents the contour at $\pm\infty$ from contributing, we find

$$\Pi(\omega)\left(1 - e^{2\pi\omega/a}\right) = 2\pi i \times (\text{Residue at } \tau = 0)$$

$$= 2\pi i \left(\frac{-a^2}{16\pi^2}\right)\left(\frac{d}{d\tau} e^{-i\omega\tau}\Big|_{\tau=0}\right) \frac{4}{a^2}$$

$$= -\frac{\omega}{2\pi}$$

Hence $\Pi(\omega) = \dfrac{\omega/2\pi}{e^{2\pi\omega/a} - 1}$, or

$$\boxed{\Pi(\omega) = \frac{|\omega|}{2\pi} \begin{cases} \dfrac{e^{2\pi|\omega|/a}}{e^{2\pi|\omega|/a} - 1} & \omega < 0 \\[2ex] \dfrac{1}{e^{2\pi\omega/a} - 1} & \omega > 0 \end{cases}}$$

(Note — we can check that this agrees with our earlier result for an inertial detector in the limit $a \to 0$.)

Now the function $\pi(\omega)$ does not vanish for $\omega > 0$. The function G_+ is strictly positive frequency according to an inertial clock, but contains both positive and negative frequency when Fourier analyzed by a uniformly accelerated clock.

If we compare the rates for excitation and deexcitation, we have

$$\frac{\text{Rate}\,(E \to E+\omega)}{\text{Rate}\,(E+\omega \to E)} = e^{-2\pi\omega/a} \quad (\text{where } \omega > 0)$$

In order to be in equilibrium with the fluctuating radiation field, the detector must occupy the states of energy E and $E+\omega$ with relative probability

$$\frac{n(E+\omega)}{n(E)} = e^{-2\pi\omega/a}$$

This is just the Boltzman distribution, with an effective value of the temperature

$$T = \frac{a}{2\pi} \quad \left(\text{or } kT = \frac{\hbar}{2\pi c}a \right.$$
$$\left. T = 4\times 10^{-23}\,{}^\circ K \times (a \text{ in cm/sec}^2) \right)$$

The detector behaves as though in contact with a thermal bath of radiation at this temperature.

To further appreciate the sense in which the uniformly accelerated observer perceives the vacuum fluctuations as thermal fluctuations, consider thermal fluctuations of a harmonic oscillator

$$x(t) = \frac{1}{\sqrt{2\omega}}\left(a e^{-i\omega t} + a^{+} e^{i\omega t}\right)$$

$$\Rightarrow \langle x(t)\, x(0)\rangle_\beta = \frac{1}{2\omega}\left\langle\left(e^{-i\omega t} a a^{+} + e^{i\omega t} a^{+} a\right\rangle_\beta\right.$$

$$= \frac{1}{2\omega}\left[e^{-i\omega t}\langle n+1\rangle_\beta + e^{i\omega t}\langle n\rangle_\beta\right]$$

$$= \frac{1}{2\omega}\left[\frac{e^{\beta\omega}}{e^{\beta\omega}-1} e^{-i\omega t} + \frac{1}{e^{\beta\omega}-1} e^{i\omega t}\right]$$

The function $\pi(\omega)$ on page (3.13) has just this form, times a density of states factor. Thus,

$$\langle 0|\phi(x(\tau))\,\phi(x(0))|0\rangle = \langle \phi(t,\vec{0})\,\phi(0)\rangle_\beta \qquad (\beta = 2\pi/a)$$

– the two-point time correlations of the field as perceived by the uniformly accelerated observer are the same as those seen by an observer at rest in a thermal state of the field, where each oscillator mode of the field has an occupation number given in expectation value by its thermal value. The higher correlations also coincide, for the $2n$-point correlations in both cases are Gaussian –

they are expressible as a product of 2pt correlations. The stochastic properties of the radiation seen by the uniformly accelerated detector are precisely those of a thermal bath.

Remarks:

- This calculation illustrates the idea discussed in chapter 2, that positive and negative frequency are observer-dependent notions. Vacuum fluctuations cannot excite the inertial detector, because they are purely positive frequency. But these fluctuations have a neg. frequency component to the accelerated observer, and so can excite his detector.

- Note that this thermal radiation does not get red shifted by a boost along $\hat{z}$ – it depends on proper acceleration, not on velocity (To the accelerated observer, the boost is just a "time" translation $\tau \to \tau +$ const.

In order to appreciate more clearly what this effect has to do with field theory in curved spacetime, we'll now rederive the thermal radiation seen by an accelerated observer using Bogolubov gymnastics – that is we'll find the Bogolubov transformation that relates the solutions that are positive frequency with respect to Minkowski time, and those that are positive (and negative) frequency with respect to the accelerated observer's clock ("Rindler time").

The coordinate system that is natural for an accelerated (Rindler) observer is one for which the world line

$$z = \frac{1}{a}\cosh a\tau$$
$$t = \frac{1}{a}\sinh a\tau$$

is at a fixed spatial position, and time is (proportional to) proper time τ. So replace z, t by ξ, η defined by

$$t := \xi \sinh\eta, \quad dt = d\xi \sinh\eta + d\eta\, \xi\cosh\eta$$
$$z := \xi\cosh\eta \quad dz = d\xi\cosh\eta + d\eta\, \xi\sinh\eta$$

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = \xi^2 d\eta^2 - d\xi^2 - dx^2 - dy^2$$

In this coordinate system $\frac{\partial}{\partial\eta}$ is timelike Killing vector (metric is η independent) — which we know is the generator of a $\hat{z}$ <u>boost</u> (a symmetry of Minkowski space). And in fact, the metric is static (the η = constant surfaces are orthogonal to the Killing vector — i.e. $g_{\eta i} = 0$)

These are the Rindler coordinates. Another way to express the coordinate transformation is in terms of light cone coordinates

$$u = t - z = -\xi e^{-\eta} = -e^{-\eta + \ln \xi} \quad (\xi > 0)$$
$$v = t + z = \xi e^{\eta} = e^{\eta + \ln \xi}$$

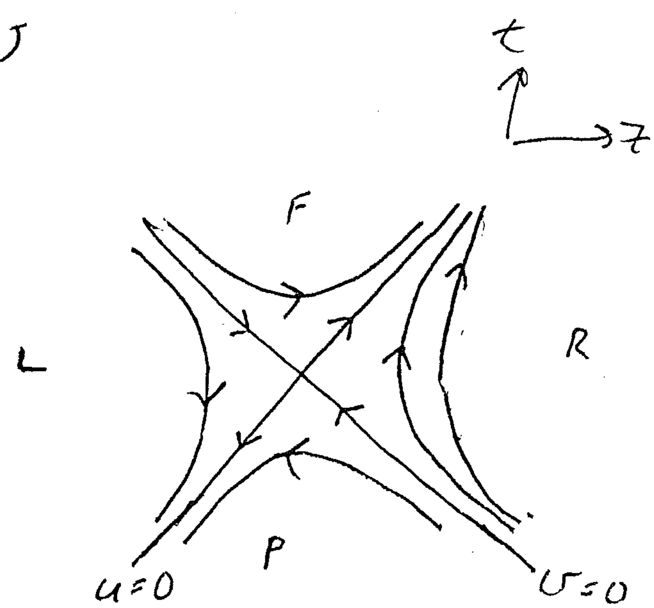
And, in fact
$$U = \eta - \ln \xi$$
$$V = \eta + \ln \xi$$

are the "light cone" coordinates for the Rindler metric, since it can be written

$$\xi^2 \left(d\eta^2 - [d(\ln \xi)]^2 \right) - dx^2 - dy^2$$

And so we have
$$u = -e^{-U}$$
$$v = e^{V}$$

Notice now, that our coordinates do not cover all of Minkowski space, but only the region



$u < 0, \ v > 0$ — Region R in the picture.

There are coordinate singularities at $u = 0$ and $v = 0$ for a good reason. We have insisted that $\partial/\partial\eta$ is a boost generator. But boosts propagate points of spacetime along hyperbolas as shown in the figure, and the tangent to the hyperbola is timelike in R and L and spacelike in F and P. So if $\partial/\partial\eta$ is a boost generator η must cross over from being timelike to spacelike on the curves $u=0$ and $v=0$.

To cover Minkowski space completely, we need to use 4 sets of Rindler coordinates in the 4 region R, L, F, P

F: $\quad t = \xi\cosh\eta \qquad u = t-z = e^{-\eta+\ln\xi}$

$u, v > 0 \quad z = \xi\sinh\eta \qquad v = t+z = e^{\eta+\ln\xi}$

$$ds^2 = -\xi^2 d\eta^2 + d\xi^2 - dx^2 - dy^2$$

L: $\quad t = -\xi\sinh\eta \qquad u = e^{-\eta+\ln\xi}$

$\quad z = -\xi\cosh\eta \qquad v = -e^{\eta+\ln\xi}$

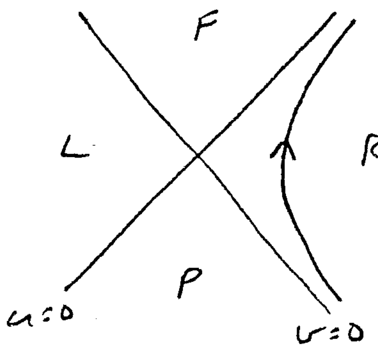
$u>0$
$v<0$

P: $\quad t = -\xi\cosh\eta \qquad u = -e^{-\eta+\ln\xi}$

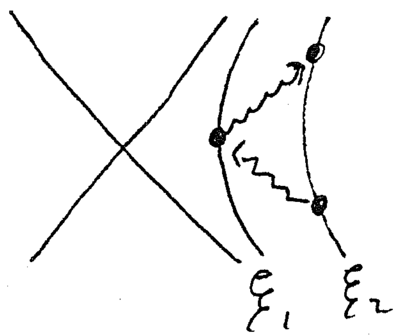
$\quad z = -\xi\sinh\eta \qquad v = -e^{-\eta+\ln\xi}$

$u, v < 0$

Now consider "causal structure" in this coordinate system.



A Rindler observer in region R (static in Rindler coordinates, and hence traveling along the hyperbola ξ = constant) can not receive any signal emitted in region F or L; hence $u=0$ is a (future) "event horizon" for this observer. From a Minkowski space viewpoint, there are null geodesics that never "catch up" with the accelerated observer. Similarly, no signal emitted from R can enter L or P, so $v=0$ is a (past) event horizon for the Rindler observer.



If two Rindler observers at ξ_1 and ξ_2 exchange signals, the signal is red (or blue) - shifted when received. The Minkowski observer attributes the shift to the ordinary special relativistic Doppler effect. E.g., we may choose an inertial frame in which the Rindler observer at ξ_1 is instantaneously at rest when he receives or emits the signal. For $\xi_2 > \xi_1$, the signal he receives is blue-shifted, and the signal he emits is red-shifted when received at ξ_2.

The light signal emitted from $(t,z) = (0, \xi_1)$ crosses the hyperbola $(t,z) = (\xi_2 \sinh\eta, \xi_2 \cosh\eta)$ at a Rindler time η given by

$$z = \xi_2 \cosh\eta = \xi_1 + t = \xi_1 + \xi_2 \sinh\eta$$

or $\xi_1/\xi_2 = \cosh\eta\,(1-\tanh\eta)$

But $\tanh\eta$ is $\frac{dz}{dt}$ - the velocity as measured by Minkowski observer of object moving on the hyperbola. The Doppler shift of the signal when received, then, is

$$\frac{\nu_2}{\nu_1} = \gamma(1-v) = \xi_1/\xi_2$$

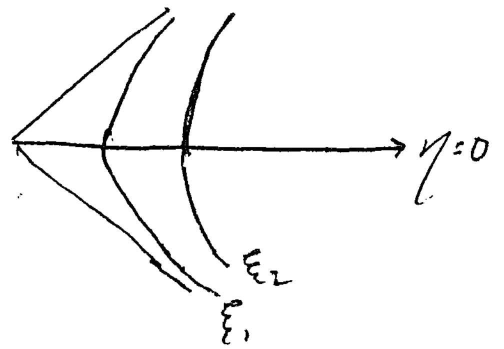
To the Minkowski observer, the shift occurs because the observer with proper acceleration $a_2 = \xi_2^{-1}$ is racing away from the point of emission when he receives the signal. Similarly, a signal emitted at ξ_2 and received at ξ_1 is blue shifted when received, because, in the frame in which receiver is at rest, the observer with proper acceleration $a_2 = \xi_2^{-1}$ was racing toward receiver as he emitted signal.

To the Rindler observers, though, the hyperbolas $\xi = \xi_1$ and $\xi = \xi_2$ are static positions. They attribute the red shift (or blue shift) to the static gravitation field

$$\frac{\nu_2}{\nu_1} = \left[\frac{g_{00}(\xi_2)}{g_{00}(\xi_1)}\right]^{-\frac{1}{2}} = \xi_1/\xi_2$$

The gravitational field is very strong, and the gravitational time delay diverges, at the past and future event horizons, where $g_{00} = \xi^2 \to 0$.

Notice that in the above discussion we have assumed that the uniformly accelerated observers with proper acceleration a_1 and a_2 are <u>simultaneously</u> at rest in the inertial frame. This is necessary because we insist that the metric is <u>static</u> when expressed in Rindler coordinates. That is, the Rindler time η has been chosen so that the uniformly accelerated observers are always moving orthogonal to the η = constant surfaces.



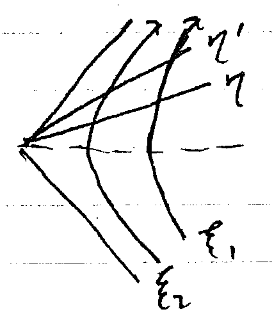
When a Rindler observer is instantaneously at rest, proper time and Minkowski time coincide, and so all Rindler observers are moving orthogonal to t = const; that is, are instantaneously at rest.

In terms of the inertial coordinate system

$$\left(\frac{\partial t}{\partial \eta}, \frac{\partial z}{\partial \eta}\right) = (\xi \cosh \eta, \xi \sinh \eta)$$

is always orthogonal to $(t, z) = (\xi \sinh \eta, \xi \cosh \eta)$ — the timelike killing vector $\frac{\partial}{\partial \eta}$ is orthogonal to the η = constant surfaces. Hence, the Rindler observers are simultaneously at rest in <u>all</u> inertial coordinate systems. (That is, velocity $dz/dt = \tanh \eta$ of static observers is same for all observers on a slice of Rindler time.)

Because the Rindler coordinate system is static, the proper distance between positions (ξ_1, x_1, y_1) and (ξ_2, x_2, y_2) is the same for all values of Rindler time η



In terms of Minkowski coordinates, the invariant interval between

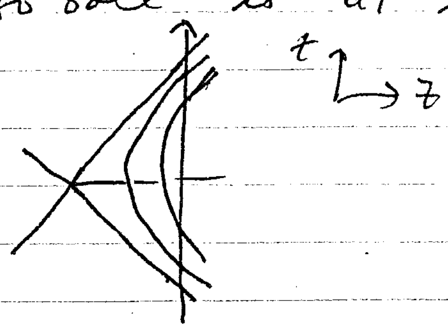
$$t_1 = \xi_1 \sinh\eta \quad \text{and} \quad t_2 = \xi_2 \sinh\eta$$
$$z_1 = \xi_1 \cosh\eta \qquad\quad\;\; z_2 = \xi_2 \cosh\eta$$

is

$$(t_1 - t_2)^2 - (z_1 - z_2)^2 = (\xi_1 - \xi_2)^2$$
$$= (\xi_1 - \xi_2)^2\left[\cosh^2\eta - \sinh^2\eta\right] = (\xi_1 - \xi_2)^2$$

– independent of η

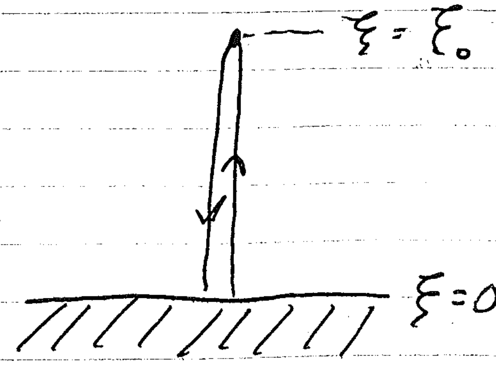
Consider a baseball moving inertially in Rindler coordinates. Choose a frame in which ball is at rest at $\eta = 0$, so ball is at rest in Minkowski coordinates.



Thus $z = \xi\cosh\eta = \text{constant}$

or $\xi = \xi_0/\cosh\eta$

To the Rindler observer, the ball pops through the floor at $\xi = 0$ at Rindler time $\eta = -\infty$, reaches its maximum height at $\eta = 0$, and then falls, reaching the floor at time $\eta = \infty$



To her, the ball is moving in a gravitational field that becomes so powerful at the $\xi=0$ floor that a light signal cannot escape from below the floor. The floor is an event horizon.

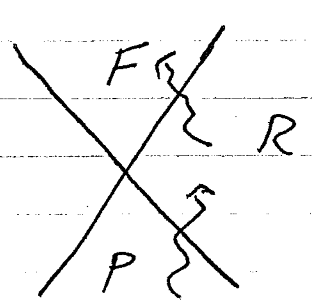
A clever Rindler physicist recognizes, however, that the ball travels from $\xi=0$ to $\xi=\xi_0$ in elapsed proper time $\tau=\xi_0$. (This is obvious to Minkowski observer.)

$$d\tau^2 = \xi^2 d\eta^2 - d\xi^2 \quad \text{and} \quad d\xi = -\frac{\xi_0 \sinh\eta}{\cosh^2\eta}\,d\eta = -\xi \tanh\eta\, d\eta$$

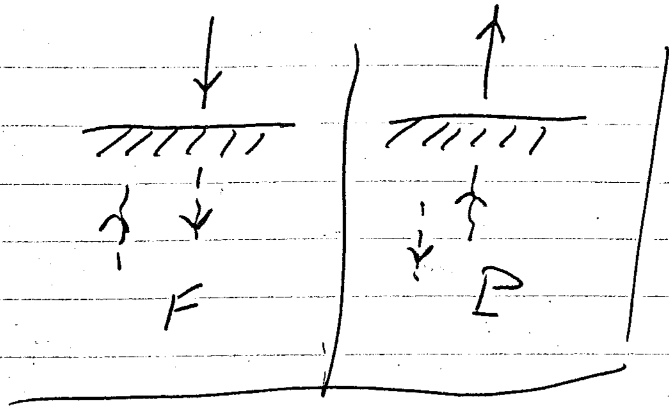
$$\Rightarrow d\tau^2 = \xi^2(1-\tanh^2\eta)\,d\eta^2 = \xi_0^2\, d\eta^2/\cosh^4\eta$$

$$\int d\tau = \xi_0\int_0^\infty \frac{d\eta}{\cosh^2\eta} = \xi_0 \tanh\eta\Big|_0^\infty = \xi_0.$$

She therefore perceives the need to extend the Rindler coordinate system beyond Rindler time $\eta=\infty$, and "before" Rindler time $\eta=-\infty$.

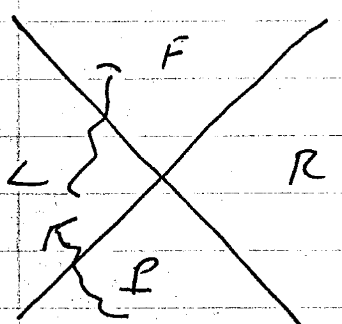


She thus discovers that there are really two horizons, in the past and future. The P region is where the ball came from and the F region is where the ball went to.

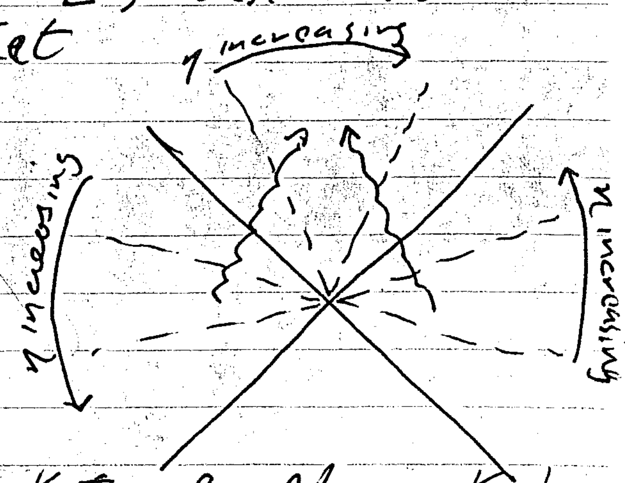


She has constructed regions of spacetime where balls moving down wind up, and where balls moving up started out.

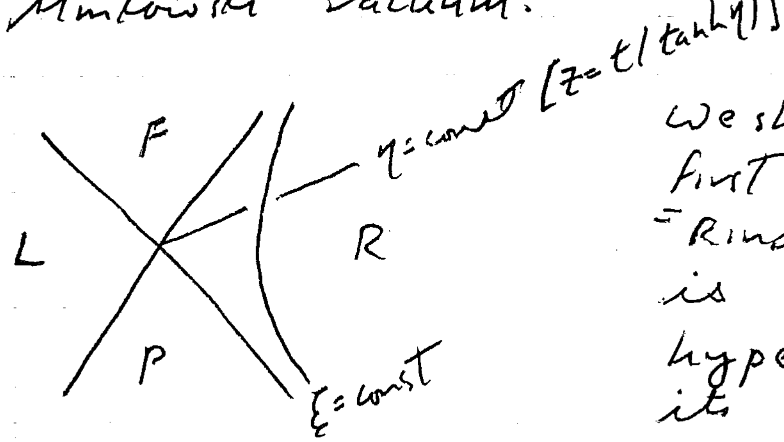
But now she recognizes that there can be balls in the F region that are going up (which had to come from somewhere) and balls in the P region going down (which have to go somewhere). She therefore discovers that, if her spacetime is geodesically complete, then it must contain a region L. She knows that L exists, even though she can neither receive signals from L or send signals to L, as long as she remains in R.



She realizes furthermore, that the Rindler time η, analytically extended to L, must run backwards. The reason is that η becomes a spacelike coordinate in region F, so that η is changing in the reverse direction for up and down moving baseballs in region F. When down and up moving timelike geodesics are extended from F back to R and L respectively, then η runs in opposite directions along the geodesics.



Now, we want to do quantum field theory on Rindler spacetime. Specifically, we will want to express the ordinary Minkowski vacuum in terms of modes that are positive frequency wrt Rindler time — in order to determine what "particles" the uniformly accelerated observer sees in the Minkowski vacuum.



We should observe first of all that the "Rindler wedge" R is a globally hyperbolic spacetime in its own right. That is, the "constant time" surface $\eta = \text{constant}$ is Cauchy in region R (All timelike or null paths through points in R cross this surface) Initial data on this surface completely determines a solution to the KG eqn throughout R. It does not suffice, of course, to determine the solution in L, F, P, but this is of no consequence to Rindler observer, as those regions are beyond his event horizon, and cannot be detected.

The same remark applies to region L. A Cauchy surface in R together with one in L is Cauchy in all of Minkowski space.

We can find a set of solutions to the KG eqn that are complete in R (or L) and are positive frequency wrt Rindler time η. In Rindler coordinates, the KG eqn (massless)

is $\partial_\mu(\sqrt{g}\; g^{\mu\nu}\partial_\nu u) = 0$

where $g_{\mu\nu} = (\xi^2, -1, -1, -1)$, $\sqrt{g} = \xi$

$g^{\mu\nu} = (\xi^{-2}, -1, -1, -1)$

So this is

$$[\partial_\eta \xi^{-1}\partial_\eta - \partial_\xi \xi \partial_\xi - \xi(\partial_x^2 + \partial_y^2)]\, u = 0$$

or $[\partial_\eta^2 - \xi\partial_\xi \xi\partial_\xi - \xi^2(\partial_x^2 + \partial_y^2)]\, u = 0$

This is easily separable, with the positive freq solns of the form

$$u = e^{-i\omega\eta} e^{i(k_x x + k_y y)} f_{k,\omega}(\xi),$$

so that

$$0 = [\xi\partial_\xi \xi\partial_\xi - (k_x^2 + k_y^2)\xi^2 + \omega^2]\, f_{k,\omega}(\xi)$$

This is Bessel's equation (for imaginary argument) and the normalizable solutions (decaying for large ξ) are (up to normalization)

$$\underbrace{f_{k,\omega}(\xi) = K_{i\omega}(k\xi)}_{\text{modified Bessel function.}} \qquad (k^2 = k_x^2 + k_y^2)$$

By normalizing these solutions, we find a complete set of positive frequency solutions

$$u^{Rin}_{R,j} \quad - \text{ to the KG eqn in the wedge } I$$

Similarly, we can construct a complete set of pos. freq solutions in region L (pos. freq wrt Rindler time)

$$u^{Rin}_{L,j}$$

— an important difference, though, is that

$$u^{Rin}_{L,j} \propto e^{i\omega\eta}$$

if η in region L is the coordinate obtained by analytically continuing η in region R, given by $t/z = \tanh\eta$

Now the u_R's and u_L's together are a complete basis for solns to the KG eqn in Minkowski space, so normalized plane waves can be expanded in terms of them. In the notation of page (2.25),

$$\begin{pmatrix} u^{Min} \\ u^{Min*} \end{pmatrix} = \begin{pmatrix} \alpha^{+} & -\beta^{T} \\ -\beta^{+} & \alpha^{T} \end{pmatrix} \begin{pmatrix} u^{Rin}_{R} \\ u^{Rin}_{L} \\ u^{Rin*}_{R} \\ u^{Rin*}_{L} \end{pmatrix}$$

or, e.g. $u_i^{Min} = (\alpha_R^+)_{ij} u_{Rj}^{Rin} + (\alpha_L^+)_{ij} u_{Lj}^{Rin}$
$- (\beta_R^T)_{ij} u_{Rj}^{Rin*} + (-\beta_L^T)_{ij} u_{Lj}^{Rin*}$

etc.

We can then evaluate Bogolubov coefficients by calculating KG inner products

$(\alpha_R^+)_{ij} = (u_i^{Min}, u_{Rj}^{Rin})$

$-(\beta_R^T)_{ij} = -(u_i^{Min}, u_{Rj}^{Rin*})$ etc.

That is, we must Fourier transform the u_R^{Rin} and u_L^{Rin}.

Fortunately, there is an easy way to get the answer (following W. Unruh). Let's first note that the Bogolubov transformation is nontrivial — the positive frequency Rindler solutions are not strictly positive frequency as Minkowski solutions. To see this, recall that an arbitrary solution that is pos freq with r.t. Minkowski time has the expansion

$$\int \frac{d^3k}{(2\pi)^3 2k^0} \tilde{f}(k) e^{-ik\cdot x}$$

and so is analytic and bounded for all $\text{Im } x^0 < 0$.

In Terms of lightcone coordinates, the Minkowski positive frequency solutions are

$$e^{-i(\omega t - K_3 z)} = e^{-\frac{i}{2}(\omega+K_3)u}\, e^{-i\frac{1}{2}(\omega-K_3)v}$$

– so we can just as well say that positive frequency solutions are analytic for $\mathrm{Im}\, u < 0$, $\mathrm{Im}\, v < 0$.

Now, in Region R ($u<0$, $v>0$)

$$\ln(-u) = -\eta + \ln\xi \qquad \Rightarrow \qquad \eta = \tfrac{1}{2}[\ln v - \ln(-u)]$$
$$\ln v = \eta + \ln\xi \qquad \ln\xi = \tfrac{1}{2}[\ln v + \ln(-u)]$$

and in Region L ($u>0$, $v<0$)

$$\ln u = -\eta + \ln\xi \qquad \Rightarrow \qquad \eta = \tfrac{1}{2}[\ln(-v) - \ln(u)]$$
$$\ln(-v) = \eta + \ln\xi \qquad \ln\xi = \tfrac{1}{2}[\ln(-v) + \ln(u)]$$

Now we can continue from Region R to Region L, remaining in domain of analyticity of Minkowski pos freq solns. But then

R: $e^{-i\eta}$ continues to L: $e^{-i\eta}$

or pos. freq. continues to neg freq wrt. Rindler time. This shows that solutions that are positive freq wrt Minkowski time must be a linear combination of pos. and neg freq wrt Rindler time.

This observation also suggests how we can construct the Bogolubov transformation. The pos. freq Minkowski solutions are the linear combinations of Rindler R and L solutions that are analytic in u (and v) in the lower half plane.

On the constant ξ hyperbolas, we have

$\eta \sim -\ln(-u)$ Region R

$\eta \sim -\ln(u)$ Region L

(Analytic continuation of v dependence gives same result as below.)

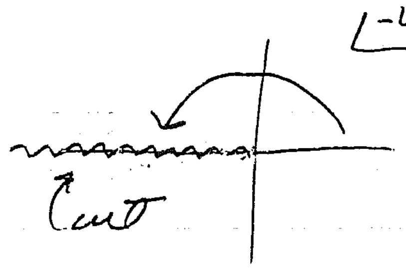
and our pos. freq Rindler solutions have the behavior

$$u^{Rin}_{R,\omega} \sim \begin{cases} e^{-i\omega\eta} \sim e^{i\omega\ln(-u)} & \text{Region R} \\ 0 & \text{Region L} \end{cases}$$

$$u^{Rin}_{L,\omega} \sim \begin{cases} 0 & \text{Region R} \\ e^{i\omega\eta} \sim e^{-i\omega\ln(u)} & \text{Region L} \end{cases}$$

Now, to get a pos freq Minkowski solution, we need to match these analytically across $u = v = 0$, where regions R and L meet; in particular, we require analyticity for $\operatorname{Im} u < 0$

Now, if we analytically continue the region R solution $\sim e^{i\omega\ln(-u)}$ (where $u < 0$) to region L, we get $e^{i\omega\ln(-u)}$ where now u is positive



But the principle that a positive freq Minkowski solution is the boundary value of a function analytic in the region $\mathrm{Im}\, u < 0$ (or $\mathrm{Im}(-u) > 0$) tells us to evaluate the Log above the cut, where

$$\ln(-u) = \ln u + i\pi$$

And so a pos. freq Minkowski solution is

$$u^{Min}_{1,\omega} = N \begin{cases} e^{i\omega \ln(-u)} & \text{Region } R \\ e^{-\pi\omega}\, e^{i\omega \ln(u)} & \text{Region } L \end{cases}$$

or

$$\boxed{u^{Min}_{1,\omega} = \left(u^{Rin}_{R,\omega} + e^{-\pi\omega}\, u^{Rin*}_{L,\omega}\right) \Big/ \underbrace{\left(1 - e^{-2\pi\omega}\right)^{\frac{1}{2}}}_{\text{normalization}}}$$

Similarly, if we continue

$e^{-i\omega \ln u}$ in Region L to

$e^{-i\omega \ln[(-1)(-u)]}$ in Region R

we must evaluate the log below the cut in the u-plane

$$\boxed{u^{Min}_{2,\omega} = \left(u^{Rin}_{L,\omega} + e^{-\pi\omega}\, u^{Rin*}_{R,\omega}\right) \Big/ \left(1 - e^{-2\pi\omega}\right)^{\frac{1}{2}}}$$

Our Bogolubov coefficients, for the solutions with Rindler frequency ω, then are

$$\begin{pmatrix} u_1^{Min} \\ u_2^{Min} \end{pmatrix} = N_\omega \begin{bmatrix} 1 & 0 & 0 & e^{-\pi\omega} \\ 0 & 1 & e^{-\pi\omega} & 0 \end{bmatrix} \begin{pmatrix} u_R^{Rin} \\ u_L^{Rin} \\ u_R^{Rin*} \\ u_L^{Rin*} \end{pmatrix}$$

$$\alpha_\omega^{+} = N_\omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$-\beta^T = N_\omega e^{-\pi\omega} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad N_\omega = \left(1-e^{-2\pi\omega}\right)^{-\frac{1}{2}}$$

Now recall our expression for the S matrix (page 2.32)

$$|0, Min\rangle = (\text{phase}) |\det \alpha|^{-\frac{1}{2}} \exp\left(\frac{1}{2} a^{Rin+} (-\beta^* \alpha^{-1}) a^{Rin+}\right) |0, Rin\rangle$$

$$-\beta^* \alpha^{-1} = e^{-\pi\omega} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

And $\det \alpha_\omega = N_\omega^2 = \dfrac{1}{1-e^{-2\pi\omega}}$

and so, up to a phase

$$|0, Min\rangle = \prod_j \left(1-e^{-2\pi\omega_j}\right)^{+\frac{1}{2}} \exp\left[\sum_j e^{-\pi\omega_j} a_{R,j}^{Rin+} a_{L,j}^{Rin+}\right] |0, Rin\rangle$$

And since $\frac{1}{n!} a_{R,j}^{Rin\dagger} a_{L,j}^{Rin\dagger} |0, Rin\rangle$

$$= |n_j, R\rangle \otimes |n_j, L\rangle$$

we have

$$|0, Min\rangle = \prod_j \left[\left(1 - e^{-2\pi\omega_j}\right)^{\frac{1}{2}} \sum_{n_j=0}^{\infty} e^{-\pi\omega_j n_j} |n_j, R\rangle \otimes |n_j, L\rangle \right]$$

Now, an observer who is static wrt Rindler time in the wedge R, and is therefore unable to observe anything in L which lies beyond his event horizon, will describe this situation in terms of a density matrix; we sum over the unobserved particles in wedge L to obtain

$$\rho_R = \sum_L |0, Min\rangle\langle 0, Min|$$

(then $\mathrm{tr}\, \rho_R \mathcal{O}_R$ gives expectation value of observable localized in region R.)

$$= \prod_j \left[\left(1 - e^{-2\pi\omega_j}\right) \sum_{n_j} e^{-2\pi\omega_j n_j} |n_j, R\rangle\langle n_j, R| \right]$$

This is exactly a density matrix (normalized to $\mathrm{tr}\, \rho = 1$) for a thermal ensemble of Rindler particles. To extract a temperature, we recall that ω is the frequency wrt the Rindler time coordinate η, and since

$$t = \xi \sinh \eta$$
$$z = \xi \cosh \eta$$

while t and z are related to the proper time τ and proper acceleration a of a Rindler observer by

$$t = \frac{1}{a} \sinh a\tau \quad \Rightarrow \quad \eta = a\tau$$
$$z = \frac{1}{a} \cosh a\tau$$

So, wrt proper time, the frequency ω becomes replaced by ω/a, and the Boltzman factor

$$e^{-2\pi\omega/a} = e^{-\omega/T} \quad \text{where } T = \frac{a}{2\pi}$$

— in agreement with our result on page 3.14, of course.

Note: T varies with Rindler position as $T = \frac{1}{2\pi\xi}$

But $(g^{00})^{\frac{1}{2}} T = \xi T = \frac{1}{2\pi}$ is a constant — Red shifted T is independent of position.

Remarks

- We have described two different ways of defining "positive frequency" in Minkowski spacetime, and two corresponding ways of defining a vacuum. Both notions of positive frequency invoke a timelike Killing vector of the spacetime. Yet we have seen that the two vacuum states are not equivalent. This happens because the two sets of positive frequency solutions are <u>not</u> related by merely analytically continuing from one time coordinate to the other.

We should recognize that the reason the two sets of pos. freq. solns are not related by analytic continuation is that Rindler spacetime has a <u>horizon</u>. It is because the Rindler Killing vector $\partial/\partial\eta$ tips over from timelike to spacelike to (reversed) timelike that pos. freq solns are pos freq wrt η in region R and neg freq wrt η in region L, and this is the origin of the nontrivial Bogolubov Transformation.

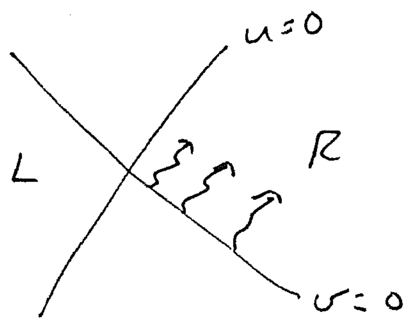
- It is worth pointing out that it is possible to derive the Bogolubov transformation by a more conventional method, and without doing a lot of work.

From $\quad u^{Rin} = \alpha u^{Min} + \beta u^{Min*}$

we have

$$\alpha_{ij} = (u_i^{Rin}, u_j^{Min}) \qquad \beta_{ij} = -(u_i^{Rin}, u_j^{Min*})$$

and evaluating these Klein-Gordon inner products amounts to Fourier transforming the Rindler solutions. These are related to Bessel functions, as noted on page (3.22).



We can simplify the integration by a felicitous choice of the surface on which we evaluate the Klein-Gordon inner product. It is convenient to choose a surface lying close to the (past) horizon $v=0$. Close to $\xi=0$, the eqn on page (3.22)

$$\left[(\xi\partial_\xi)^2+\omega^2-k_\perp^2\xi^2\right]f_{k,\omega}(\xi)=0$$

simplifies — for the $k_\perp^2\xi^2$ term can be neglected and the KG solutions of positive frequency have the form (up to normalization)

$$e^{-i\omega\eta}e^{\pm i\omega\ln\xi}e^{ik_\perp\cdot(x,y)}$$

$$=\begin{Bmatrix}e^{-i\omega U}\\ e^{-i\omega V}\end{Bmatrix}e^{ik_\perp\cdot(\vec{x},\vec{y})}\quad\text{in region } R,$$

where

$$u=t-z=-e^{-U}$$
$$v=t+z=e^{V}$$

What is happening is that the KG solutions of definite Rindler frequency become strongly blue shifted as they approach the event horizon, so that the transverse wavenumbers become negligible, and the problem reduces to an effectively two-dimensional one

The solutions of definite Minkowski frequency are

$$e^{-i\omega t} e^{ik_3 z} e^{ik_\perp \cdot (\vec{x}, \vec{y})}$$

On the surface $v=0$ $\quad t = \frac{1}{2}(u+v) = \frac{1}{2}u$

$\quad z = \frac{1}{2}(v-u) = -\frac{1}{2}u$

So we have $e^{-i\frac{1}{2}(\omega + k_3)u} \; e^{ik_\perp \cdot (\vec{x}, \vec{y})}$

Now, on the surface $v=0$, we can expand the positive frequency Rindler solutions in terms of positive and negative frequency Minkowski solutions. Consider

$$f(u) = \begin{cases} e^{i\omega \ln(-u)} & u<0 \\ 0 & u>0 \end{cases}$$

If we Fourier transform,

$$f(u) = \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{-i\sigma u} \tilde{f}(\sigma).$$

And you may show (exercise)

$$\tilde{f}(-\sigma) = -e^{-\pi\omega} \tilde{f}(\sigma) \;, \text{ for } \sigma>0$$

Thus

$$u^{Rin}_{R,\omega} = N_\omega \left(u^{Min} - e^{-\pi\omega} u^{Min *} \right)$$

— the Bogolubov transformation again

- It is frequency wrt Rindler time η that appears in the Boltzman factor $e^{-2\pi\omega}$. This needs to be translated into frequency with respect to proper time of an observer fixed in Rindler coordinates, in order to determine what energy state of a uniformly accelerated detector gets excited.

Since $d\tau^2 = \xi^2 d\eta^2$

$\frac{d}{d\tau} = \frac{1}{\xi} d/d\eta$ and so proper time frequency is $\omega/\xi \Rightarrow$ we have $e^{-2\pi\xi\omega_{proper}}$

So $T = \frac{1}{2\pi\xi}$

Thus, as a detector moves to smaller ξ it gets hotter. This is thermal equilibrium though, because radiation gets blue shifted as it moves to smaller ξ.

$$\frac{T_2}{T_1} = \left[\frac{g_{00}(\xi_2)}{g_{00}(\xi_1)}\right]^{-\frac{1}{2}} = \frac{\xi_1}{\xi_2}$$

What does the Minkowski observer see when the accelerating detector absorbs a quantum from the Rindler thermal gas? Since absorption changes the state of the radiation field, and the state is initially the Minkowski vacuum, he must see the emission of a Minkowski quantum by the detector. If we invert the Bogolubov transformation as on page (2.25), we have

$$a_R^{Rin} = (\alpha_R^* a^{Min} - \beta_R^* a^{Min\,+})$$
$$= (a_{1,\omega}^{Min} + e^{-\pi\omega} a_{2,\omega}^{Min\,+}) / (1-e^{-2\pi\omega})^{\frac{1}{2}}$$

(Remember, the Minkowski quantum created by $a_{2,\omega}^{Min}$ does not have definite Minkowski frequency). In the Minkowski Fock basis then, absorption of the Rindler quantum excites the Minkowski vacuum

$$|0, Min\rangle \to a_R^{Rin} |0, Min\rangle$$
$$= \frac{e^{-\pi\omega}}{(1-e^{-2\pi\omega})^{\frac{1}{2}}} a_{2,\omega}^{Min\,+} |0, Min\rangle$$

oddly, the mode function of this quantum that is created when a detector absorbs a Rindler quantum in region R is

$$u_{2,\omega}^{Min} = \frac{1}{(1-e^{-2\pi\omega})^{\frac{1}{2}}} (e^{-\pi\omega} u_{R,\omega}^{Rin*} + u_{L,\omega}^{Rin})$$

which is predominantly localized in region L.

Now, the energy of the absorbed quantum is seen by the Rindler observer to be

$$\frac{1}{\xi}\omega = \omega_{proper} = \omega a \quad \text{where}$$
a is the accelerating the detector