The Rindler energy is

$$E_{Rin} = \int_{\Sigma} n^\mu n^\nu T_{\mu\nu}\, d^3x$$

where the integral is over a surface Σ of constant Rindler time η, and n^μ is the unit normal to the surface. It is (up to the rescaling by $\frac{1}{\xi}$) just the generator $i\frac{\partial}{\partial\eta}$ of a translation of Rindler time

Now, both the Minkowski and Rindler observers can calculate the change in E_{Rin} stored in the field when the detector becomes excited, although of course the Rindler observer can see only $T_{\mu\nu}$ in region R. The Minkowski observer sees a change in E_{Rin} in region R

$$\delta E_{Rin} = \left(u^{Min}_{Z,\omega}, \left(\frac{1}{\xi} i \frac{\partial}{\partial\eta}\Big|_{\text{region } R} u^{Min}_{Z,\omega} \right) \right) \Big/ \left(u^{Min}_{Z,\omega}, u^{Min}_{Z,\omega}\right)$$

– recalling the isomorphism relating KG solutions and the oneparticle Hilbert space

$$\delta E_{Rin} = \frac{e^{-2\pi\omega}}{1-e^{-2\pi\omega}} \left(u^{Rin*}_{R,\omega}, \frac{1}{\xi} i \frac{\partial}{\partial\eta} u^{Rin*}_{R,\omega} \right)$$

$$= \frac{e^{-2\pi\omega}}{1-e^{-2\pi\omega}} \,(\omega/\xi)$$

$$= e^{-\beta E} \Big/ 1-e^{-\beta E}\,(E) = \frac{E}{e^{\beta E}-1}$$

where E is the energy of the detected particle and β is the local temperature where it is detected.

Consistency requires that the Rindler observer also believes that absorption of energy-E quantum has increased the energy of the radiation bath by

$$E/(e^{\beta E}-1)$$

But this is so! From the point of view of the Rindler observer, one quantum of energy E has been removed from a particular field mode. Surely he must think that the energy of this mode has gone down, not up.

NO! In thermal equilibrium, the number of quanta populating this field mode has the probability distribution

$$P_n = (1-e^{-\beta E})\, e^{-\beta n E}$$

and the mean occupation number is

$$\langle n(E)\rangle_\beta = \frac{1}{e^{\beta E}-1}$$

But this probability distribution is <u>different</u> after we detect a particle in the mode, because we are more likely to have found a particle there if the mode is highly occupied

After detection, we have

$$\underbrace{P_{n-1}}_{n-1 \text{ because one quantum has been removed}} \propto n e^{-\beta n E}$$

With this new probability distribution, one finds (exercise)

$$\sum_n n P_n = \frac{2}{e^{\beta E}-1} = 2\langle n(E)\rangle_\beta$$

— the mean occupation number has gone up, even though a quantum has been removed. Indeed

$$\delta n = \langle n(E)\rangle_\beta \quad \text{and} \quad \delta E = E\langle n(E)\rangle_\beta$$

— in exact agreement with what the Minkowski observer saw. So — the Minkowski observer thinks that the energy in the field went up because the accelerating detector emitted a quantum in the Minkowski vacuum. The Rindler observer thinks that the energy went up (by the same amount) because the detector absorbed a quantum from the thermal Rindler gas.

(This remarkable observation is due to W. Unruh and R. Wald, Phys. Rev. D 29 (1984) 1047.

If $\langle E \rangle$ actually goes up when a particle is removed from the thermal gas, can we continue to extract energy indefinitely? Surely not.

We implicitly assumed that the detector is weakly coupled to the bath, so that it interacts only once while detector and gas are coupled. Suppose

$$\epsilon n$$

is probability that detector gets excited while in contact, where n is the no. of quanta in the mode, and $\epsilon \ll 1$. (This is prob to lowest order in ϵn.) Then there is probability

$$(1-\epsilon n)$$

that no detection occurs. A failure to detect __lowers__ $\langle n \rangle$, for we have

$$P_n \propto (1-\epsilon n) e^{-n\beta E}.$$

You can check (exercise) that, when the possibility of no detection is taken into account, measurements do not, on the average, increase $\langle E \rangle$.

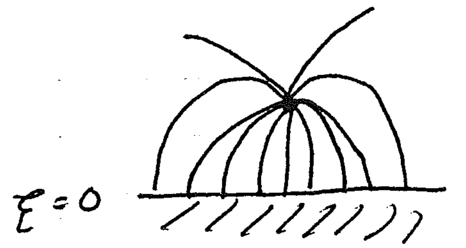
Note: the Minkowski observer agrees with the Rindler observer that the detector has become excited. To her, the energy of excitation and of the emitted quantum are supplied by the agent that is accelerating the detector. She might say that the emitted radiation has exerted a radiation reaction force on the detector that has kicked it to an excited state.

Aside: Classical radiation from uniformly accelerated charge

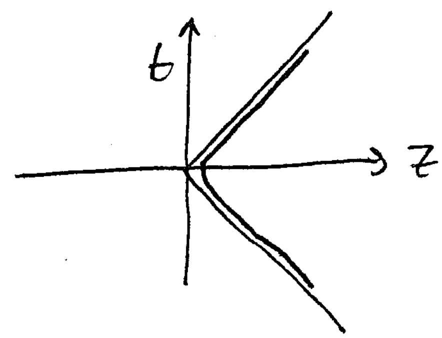
This doesn't have much to do with QFT, but just for fun, we'll briefly consider the classical em radiation emitted by a uniformly accelerated charge, from the perspective of Minkowski and Rindler observers.

(For details, see D. Boulware, Ann. Phys. 124 (1980) 169)

To an observer moving inertially in Minkowski space, a uniformly accelerated charged particle will surely radiate. But to the Rindler observer, the charge is merely at rest in a gravitational field; surely it does not radiate. How do we reconcile these two points of view?

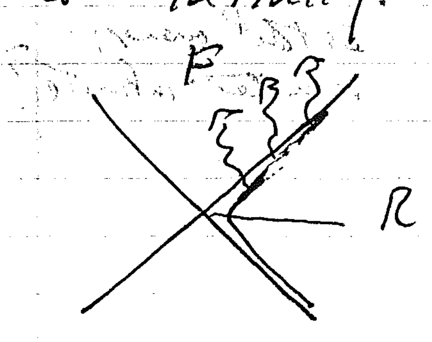


By the way, the gravitational field does distort the Coulomb field of the charge. This distortion can be calculated by noting that the electric field must be normal to the horizon at $\xi = 0$, as measured by static Rindler observers. This is necessary so that an inertial observer will not see a divergent E field at the horizon



(a boost in the $\hat{z}$ direction changes the transverse field as $E_{x,y} \to \gamma E_{x,y}$).
Thus, the horizon behaves like a conducting surface. (see "The Membrane Paradigm" by Thorne, Price, and MacDonald.)

To identify emitted radiation, we must distinguish "near field" effects from distortions of the field that propagate to infinity.



The key to reconciling the two viewpoints is that all the emitted radiation seen by the Minkowski observer leaves the Rindler wedge R and propagates into F (except for that emitted precisely in the $\hat{z}$ (forward) direction). The field of the moving charge is a static Coulomb field in region R. There is a radiation field in region F, but this is beyond the Rindler horizon.

There is another puzzle, though. We expect a radiating charge to feel a radiation reaction force. How does the Rindler observer interpret such a force exerted on a static charge?

The answer is that there is no radiation reaction force, even to the Minkowski observer, for uniformly accelerated charge.

Now we seem to have a problem with energy conservation, from the Minkowski observers viewpoint. Where is the energy that is being radiated away coming from? (Since there is no radiation reaction, the work done by the agent that accelerates the charge is used to change its momentum, with nothing left over to account for emitted radiation.)

The energy that is radiated away actually comes from the change in the self energy of the field of the charge. (This self energy does not depend only on velocity, for accelerated charge.) If we considered a charge that moved inertially before and after a period of uniform acceleration, then the work done by radiation reaction would equal the energy radiated.

Note that the thermal radiation that we have discovered is seen by a Rindler observer is a quantum-mechanical effect, with Temperature

$$T = a\hbar/2\pi c$$

that approaches 0 in the small $\hbar$ limit

Returning now to this quantum effect, we take note of the very strong correlations in the state $|0, M_{in}\rangle$ between Rindler modes in the R and L wedges. We had

$$|0, M_{in}\rangle = \prod_j N_j \sum_{n_j=0}^{\infty} e^{-\pi\omega_j n_j} |n_j; R\rangle \otimes |n_j; L\rangle.$$

Thus, by detecting a Rindler particle in region R, we can learn something about the state of the field in the causally disconnected region L.

These are EPR-like correlations, and although they exist, they do not provide a mechanism for acausal communication. Causality, as usual, is enforced by the condition $[\phi(x), \phi(y)] = 0$ for spacelike separation, which is rigorously satisfied.

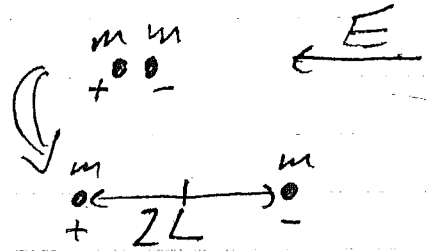
The existence of field correlations over spacelike separation we have encountered before. E.g.,

we have $\langle 0|\phi(x)\phi(y)|0\rangle = 0$ for $(x-y)^2 < 0$
so outcomes of field measurements at x and y are correlated for spacelike separation. But these correlations do not allow observers making measurements at x and y to send information to one another. (Cf. EPR)

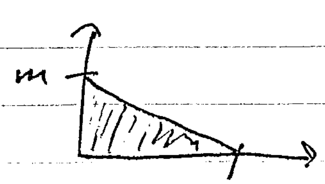
Tunneling Interpretation

Here is a heuristic interpretation of the radiation seen by a Rindler observer:

Compare dielectric breakdown of vacuum. In a constant electric field, it is energetically allowed for a particle antiparticle pair to be produced. The cost



$2mc^2$ of making pair is regained by separating them by distance $2L$, where
$$2m = eE(2L)$$



Each particle must penetrate a barrier of height m and width L. The amplitude for barrier penetration can be estimated semiclassically by solving KG eqn for charged particle in an electric field by WKB.

This eqn is $[D_\mu D^\mu + m^2]\,u(x) = 0$

where

$$D_\mu = (\partial_\mu + ieA_\mu) \text{ is gauge-covariant derivative}$$

and $\quad A_0 = Ez$ for $\vec{E} = E\hat{z}$

(in a particular gauge). So

$$(\partial_0 + ieEz)^2 u = (\vec{\nabla}^2 - m^2)\,u$$

So $u = e^{-i\omega t}\hat{u}(\vec{x})$

$$\Rightarrow \left[\vec{\nabla}^2 - m^2 + (\omega - eEz)^2\right]\hat{u}(\vec{x})$$

Now consider zero energy ($\omega = 0$) solution propagating through the barrier

In WKB approximation, the solution is

$$\hat{u} \sim \exp\left[-\tfrac{1}{\hbar}\int dz\; K\right]$$

$$K^2 = -m^2 + (eEz)^2$$

or, we get a tunneling amplitude

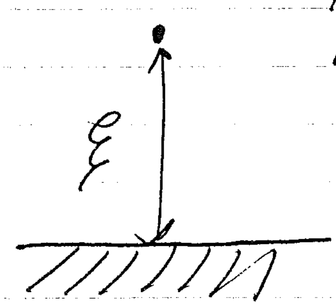
$$\exp\left[-\frac{1}{\hbar}\int_0^L dz\,\sqrt{m^2 - (eEz)^2}\right]$$

$$= \exp\left[-\frac{m^2}{\hbar eE}\int_0^1 dx\,\sqrt{1-x^2}\right] = \exp\left[\frac{-\pi}{4}\frac{m^2}{\hbar eE}\right]$$

Squaring the amplitude and including same factor for tunneling of the antiparticle, we obtain

$$\text{Rate} \sim \exp\left[-\frac{\pi m^2 c^3}{\hbar e E}\right]$$

— to be interpreted as a rate per unit time and volume of pair production.



Now consider the Rindler case. We would like to think of Rindler particles as originating due to pair production in the strong gravitational field. And indeed, when we expand $|0, \text{Min}\rangle$ in terms of Rindler modes, we always get <u>pairs</u> of particles, one in wedge L and one in wedge R. In a formal sense, each pair has vanishing Rindler energy, because the particles in wedges R and L have <u>opposite</u> values of $i\frac{\partial}{\partial \eta}$, as Rindler time η runs <u>backwards</u> in Region R.

Loosely, in analogy with dielectric breakdown, this pair production is a tunneling process, where, for a quantum with energy E, the barrier has height E and width ξ, the proper distance to the horizon. So there is a tunneling factor of the form

$$\exp\left[-\frac{1}{\hbar}\xi E\right]$$

This is suggestive of a Boltzman factor with an effective Temperature

$$T \sim \hbar/\xi .$$

Is there a more systematic formulation of this argument that would, say, account for the numerical factor $1/2\pi$?

Thermodynamics

On first acquaintance, it seems remarkable that an accelerated observer should see a thermal bath of radiation, even if it is not so remarkable that he detects quanta in the Minkowski vacuum. What does uniform acceleration have to do with Thermodynamics?

The "tunneling" argument above suggests a thermal distribution, as the tunneling factor $\sim e^{-E}$. This argument also suggests that the existence of a horizon is important — it is the existence of a horizon that ensures that the "width of the barrier" is a universal distance independent of energy. (cf the case of dielectric breakdown, for which the barrier is wider for a heavier particle.)

Note also that it is important that a uniformly accelerated observer sees a static spacetime, so it is possible in principle to talk about equilibrium thermodynamics.

Here is another attempt to "explain" why the Minkowski vacuum should look like a thermal bath to an accelerated observer:

First recall that, because of the horizon, a pure quantum state from the Minkowski observer's viewpoint will appear to be a mixed state to the accelerated observer. This is because the "Rindler observables" are localized in wedge R, but e.g. the Minkowski vacuum establishes correlations between fields in R and fields in L. Expectation values of Rindler observables appear to be evaluated in a mixed state density matrix that is obtained by summing with respect to the unobserved state of the fields in L.

But why a thermal state, a very special kind of mixed state? It must be because the Minkowski vacuum is a very special kind of pure state.

One way of characterizing how the vacuum is special is — it is stable with respect to small perturbations. E.g.

consider a free field theory, an assembly of uncoupled harmonic oscillators. And imagine introducing a small perturbation that couples the modes together while preserving the total energy (that is, the perturbation H' commutes with the unperturbed energy H_0.) Because the vacuum is a nondegenerate eigenstate of H_0, and H and H' can be simultaneously diagonalized, it remains an eigenstate (and in fact the ground state) of $H = H_0 + H'$.

But there are no other states that remain eigenstates (and hence time independent) for arbitrary perturbations H' such that $[H', H_0] = 0$, because the vacuum is the only nondegenerate eigenstate of H_0.

Furthermore, because of its nondegeneracy, the ground state of H_0 is in a sense stable wrt arbitrary small perturbations. For this state ψ_0 remains nearly time independent in the sense that

$$|(\psi_0(t), \psi_0(0))| \sim 1 \quad \text{where } \psi_0(t) = e^{-iHt}\psi_0(0)$$

for all t, and for any small H'. $H = H_0 + H'$

(This is not true of the excited states for arbitrary perturbations, even if H' and H_0 do commute, for energy may "flow" from one mode to another.)

These considerations suggest that the mixed state that the Rindler observer sees when he looks at the Minkowski vacuum should also have the property of being a state that is static with respect to Rindler time and has the property of being stable to turning on of arbitrary small perturbations that couple the Rindler modes. But a state with this property is precisely a thermal mixed state at some temperature! It is the fundamental principle of statistical mechanics that the thermal state is the unique state that remains stable when the microscopic degrees of freedom are coupled together in an arbitrary fashion.

(An argument something like this appears in D. W. Sciama, P. Candelas and D. Deutsch, Adv. in Phys. 30 (1981) 327.)

We can understand the emergence of thermodynamics in yet another way, by considering the properties of time correlations in thermal equilibrium. We will see that there is a good reason for the Rindler observer to see these characteristic correlations.

Time Correlations in Thermal Equilibrium

Consider one-dimensional harmonic oscillator

$$x(t) = \frac{1}{\sqrt{2\omega}}\left[e^{-i\omega t}a + e^{i\omega t}a^\dagger\right]$$

(cf page 3.15). Then

$$G_+^\beta(t) = \langle x(t)\, x(0)\rangle_\beta$$

$$= \frac{1}{2\omega}\langle e^{-i\omega t}a a^\dagger + e^{i\omega t}a^\dagger a\rangle_\beta$$

$$= \frac{1}{2\omega}\left[\langle n+1\rangle_\beta e^{-i\omega t} + \langle n\rangle_\beta e^{i\omega t}\right]$$

$$= \frac{e^{\beta\omega}}{2\omega(e^{\beta\omega}-1)}\left(e^{-i\omega t} + e^{-\beta\omega}e^{i\omega t}\right)$$

$$\frac{\langle n\rangle_\beta}{\langle n\rangle_\beta + 1} = e^{-\beta\omega}$$

(Boltzman)

Note that the function

$$f_+^\beta(t) = e^{-i\omega t} + e^{-\beta\omega}e^{i\omega t}$$

has the property

$$f_+^\beta(t - i\beta) = e^{-\beta\omega}e^{-i\omega t} + e^{i\omega t}$$

$$= f_+^\beta(-t)$$

So, the correlation function $G_+^\beta(t)$, when analytically continued to complex t plane, has the property

$$G_+^\beta(t-i\beta) = G_+^\beta(-t) \equiv G_-^\beta(t)$$

or $$\langle X(t-i\beta)\,X(0)\rangle_\beta = \langle X(0)\,X(-t)\rangle_\beta$$

This property (and its generalizations) is called the Kubo-Martin-Schwinger (KMS) condition - a fundamental property of correlation functions in statistical mechanics.

The KMS property is much more general than this example — It can be extended to arbitrary operators and Hamiltonians. Consider

$$\langle A(t)\,B(0)\rangle_\beta$$

where $A(t) = e^{iHt}A(0)e^{-iHt} \equiv e^{iHt}Ae^{-iHt}$ is Heisenberg operator and

$$\langle \mathcal{O}\rangle_\beta = \frac{1}{Z}\,\mathrm{tr}(e^{-\beta H}\mathcal{O}),\quad Z = \mathrm{tr}\,e^{-\beta H},$$

is thermal expectation value. Note first that

$$\mathrm{tr}(e^{-\beta H}e^{iHt}Ae^{-itH}B)$$
$$= \mathrm{tr}(e^{-\beta H}Ae^{-itH}Be^{iHt})$$

since trace is cyclic, and $e^{-\beta H}$, e^{iHt} commute

Thus $\langle A(t)\,B(0)\rangle_\beta = \langle A(0)\,B(-t)\rangle_\beta$; thermal equilibrium respects time-translation invariance.

Also

$$\frac{1}{Z}\,\mathrm{tr}\left(e^{i(t+i\beta)H} A\, e^{-itH} B\right) = \langle A(t) B(0) \rangle_\beta$$

has the formal property

$$\langle A(t-i\beta) B(0) \rangle_\beta = \frac{1}{Z}\,\mathrm{tr}\left(e^{itH} A\, e^{i(-t+i\beta)} B\right)$$

$$= \frac{1}{Z}\,\mathrm{tr}\left(e^{i(-t+i\beta)} B\, e^{itH} A\right)$$

$$= \langle B(-t) A(0) \rangle_\beta$$

– KMS condition

Consider now the special case of a free scalar field, and pursue the analytic continuation into the complex t plane in greater detail.

The free field is just a superposition of harmonic oscillators, so the thermal correlation function

$$G_+^\beta(t,\vec{x}) = \langle \phi(t,\vec{x})\, \phi(0) \rangle_\beta$$

is given by a sum over the modes of the field. Considered as a function of t for $\vec{x}$ fixed, we have

$$G_+^\beta(t) \propto \sum_{\text{modes } k} \left(e^{-i\omega_k t} + e^{i\omega_k t} e^{-\beta\omega_k}\right)$$

For $\beta \to \infty$ (zero temperature) this is the positive frequency function

$$G_+(t) \propto \sum_k e^{-i\omega_k t}$$

that we have considered before; the sum converges and $G_+(t)$ is an analytic function for $\mathrm{Im}\, t < 0$. There is also a function

$$G_-^\beta(t,\vec{x}) = \langle \phi(0)\, \phi(t,\vec{x}) \rangle_\beta = G_+^\beta(-t,-\vec{x})$$

$$= G_+^\beta(-t,\vec{x})$$

(rotational invariance),

which is purely negative frequency for $\beta \to \infty$

$$G_-(t) \propto \sum_k e^{i\omega_k t};$$

this is analytic for $\mathrm{Im}\, t > 0$.

For finite β, the analytic structure is modified. For

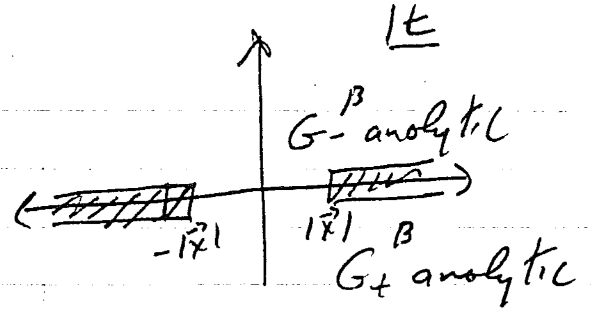
$$G_+^\beta \propto \sum_k \left(e^{-i\omega_k t} + e^{i\omega_k t} e^{-\beta\omega_k} \right)$$

is a convergent sum only in a <u>strip</u>

$$-\beta < \mathrm{Im}\, t < 0,$$

and G_-^β converges only for

$$0 < \mathrm{Im}\, t < \beta .$$



Furthermore, we know that fields commute for spacelike separation. So for $|\vec{x}| \neq 0$, there is an interval

$$-|\vec{x}| < t < |\vec{x}|$$

on the real t axis where $G_-^\beta(t) = G_+^\beta(t)$. Thus $G_-^\beta(t)$ is the analytic continuation on the upper strip of the function $G_+^\beta(t)$ that is analytic on the lower strip. Evidently, there are cuts beginning $t = |\vec{x}|$, since

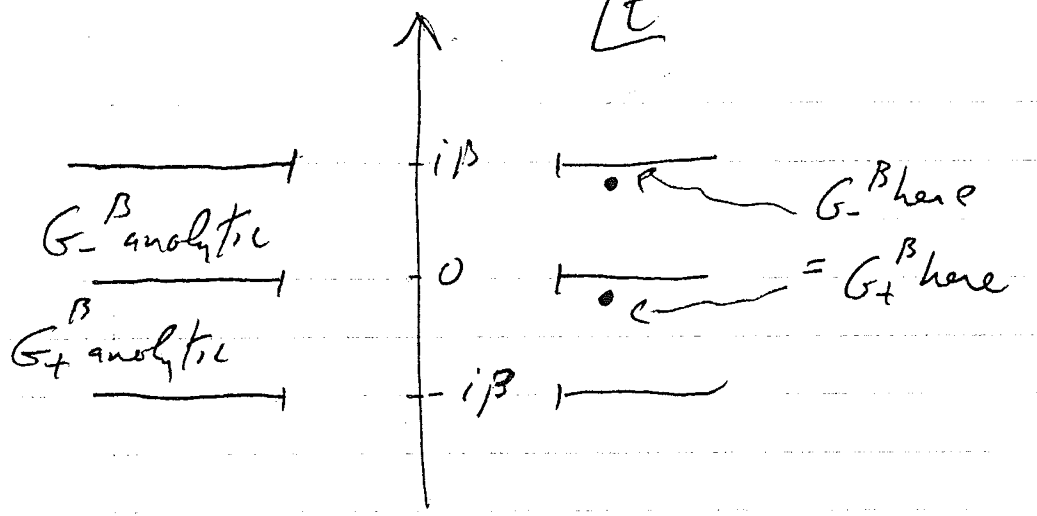
$$G_+^\beta(t,\vec{x}) - G_-^\beta(t,\vec{x}) = \langle [\phi(x), \phi(0)] \rangle_\beta$$
$$= i\, G^{\beta=0}(x)$$

We may interpret G_+ and G_- as the values of an analytic function in the cut t plane, as the cut is approached from below and above respectively.

Now recall the KMS condition, which in this case tells us

$$G_+^\beta(t - i\beta) = G_-^\beta(t)$$

From this . . . . and the property noted above, we can see how to analytically continue G_+^β beyond the strip, to the whole complex t plane.



The KMS condition says that our analytic function is periodic with period $i\beta$, and so it can be periodically extended to all values of $\mathrm{Im}\, t$

We find then, that thermal correlation functions are given by real boundary values of an analytic function in the complex t plane that is periodic in imaginary time with period β. The result is evidently quite general — it applies to arbitrary field operators in an <u>interacting</u> theory, for KMS is satisfied for such correlators, and we can continue across the real t axis as long as the fields commute for spacelike separation (are local observables)

Note: It is often said that the time ordered thermal Green functions

$$iG_F^\beta(t) = \theta(t)\, G_+^\beta(t) + \theta(-t)\, G_-^\beta(t)$$

have this periodicity property. But evidently iG_F^β is just a boundary value of this same analytic function, where the cut is approached from below for $t>0$ and from above for $t<0$.

Of course, since the Riemann surface of this function has various sheets, we need to specify what sheet we are on in discussing periodicity properties. In the above discussion, we considered continuing <u>around</u> the cuts, but the structure is different if we continue under the cuts.

A useful way to characterize the analytic function that we have constructed, in either the $T=0$ or finite T case, is that it is an (analytically continued) Green function for the Klein-Gordon equation. Let's consider the function $G(x)$, in the $T=0$ case, that matches $G_+(x)$ in the $\text{Im}\, t<0$ LHP and matches $G_-(x)$ in the $\text{Im}\, t>0$ UHP. On the imaginary axis $t=i\tau$, this function has no singularity for $|\vec{x}|\neq 0$. By specifying $G(x)$ for $t=i\tau$, we determine it throughout the cut plane, by the uniqueness of analytic continuation.

In fact, $G(x)$ for $t=i\tau$ is the <u>unique</u> Green function for the <u>Euclidean</u> Klein-Gordon equation (in flat space)

$$\left(\frac{\partial^2}{\partial\tau^2}+\vec{\nabla}^2-m^2\right)\phi(\tau,\vec{x})=f$$

Since $\phi = 0$ is the unique solution to

$$(\Box_E - m^2)\,\phi = 0$$

that vanishes at Euclidean infinity, the Euclidean K-G operator

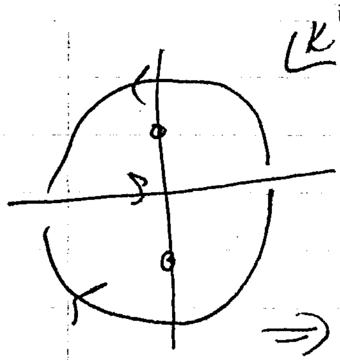
$$\Box_E - m^2 \equiv \frac{\partial^2}{\partial\tau^2} + \vec{\nabla}^2 - m^2$$

is <u>invertible</u>. (<u>unlike</u> the Minkowski K-G operator.) The unique decaying solution to

$$(\Box_E - m^2)\, G_E(\tau, \vec{x}) = -\delta(\tau)\,\delta^3(\vec{x})$$

may be expressed as

$$G_E(\tau, \vec{x}) = \int \frac{d^4k}{(2\pi)^4}\, e^{i\vec{k}\cdot\vec{x}}\, e^{ik^0\tau}\, \frac{1}{(k^0)^2 + \vec{k}^2 + m^2}$$



Now, do the k^0 integral by completing the contour in the $\begin{cases} \text{UHP for } \tau > 0 \\ \text{LHP for } \tau < 0 \end{cases}$

$$\Rightarrow G_E(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}} \begin{cases} \dfrac{i}{2i\omega_k}\, e^{-\omega_k\tau} & \tau > 0 \\ \dfrac{-i}{-2i\omega_k}\, e^{\omega_k\tau} & \tau < 0 \end{cases}$$

$$= \int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\, e^{i\vec{k}\cdot\vec{x}}\, e^{-\omega_k|\tau|}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$

this evidently agrees with

$$G_+(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i\vec{k}\cdot\vec{x}} e^{-i\omega_k t}$$

when continued to $t = i\tau$ $\tau < 0$,
and with

$$G_-(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i\vec{k}\cdot\vec{x}} e^{i\omega_k t}$$

when continued to $t = i\tau$, $\tau > 0$,
just as we asserted

A similar construction allows us to relate the function G^β to a Euclidean KG Green function, but now it is a Green function that obeys the cylinder boundary condition — i.e. is periodic in τ under

$$\tau \to \tau + \beta$$

– this Green function inverts $\Box_E - m^2$ on the cylinder.

The unique decaying solution to

$$(\Box_E - m^2)\, G_E^\beta(\tau,\vec{x}) = -\delta(\tau)\,\delta^3(\vec{x})$$

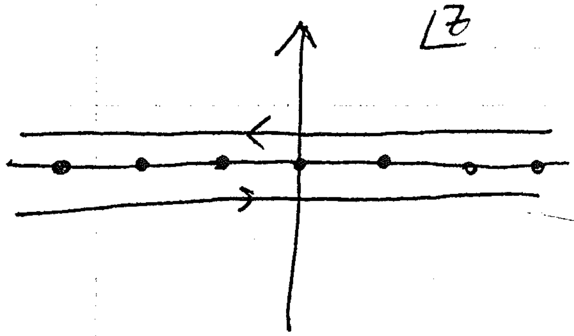
may be expressed as

$$G_E^\beta(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\beta} \sum_{n=-\infty}^{\infty} e^{i\vec{k}\cdot\vec{x}} e^{i\frac{2\pi}{\beta}n\tau} \frac{1}{\left(\frac{2\pi n}{\beta}\right)^2 + \vec{k}^2 + m^2}$$

since $\quad \frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi}{\beta}n\tau} = \delta(\tau)$

on the interval $\quad \tau \in [0,\beta]$

We wish to verify, by doing the sum over n explicitly, that this Euclidean Green function agrees with our expressions for G_+^β on lower strip and G_-^β on upper strip. Then the function we constructed earlier is the analytic continuation of the Green function G_E^β off the $t = i\tau$ axis.



Here is a trick for doing such sums (the Sommerfeld-Watson transform) — we convert it to a contour integral.

$$\sum_{n=-\infty}^{\infty} f\left(\frac{2\pi}{\beta}n\right) = \frac{1}{2\pi i}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} dz\, f(z)\, \frac{\beta}{2}\cot\left(\frac{\beta z}{2}\right)$$

$$+ \frac{1}{2\pi i}\int_{\infty+i\epsilon}^{-\infty+i\epsilon} dz\, f(z)\, \frac{\beta}{2}\cot\left(\frac{\beta z}{2}\right)$$

— since $\cot\left(\frac{\beta z}{2}\right)$ has poles at $z = \frac{2\pi}{\beta}n$, each with residue $= 1$. This trick is useful if we can simplify evaluation of the integral by distorting the contour.

In the case of interest, it is convenient to write

$$\sum_{n=-\infty}^{\infty} f\left(\frac{2\pi}{\beta}n\right) = \frac{\beta}{2\pi}\int_{-\infty}^{\infty} dz\, f(z)$$

$$+ \frac{1}{2\pi i}\int_{-\infty-it}^{\infty-it} dz\ f(z)\ \frac{\beta}{2}\left(\cot\frac{\beta z}{2} - i\right)$$

$$+ \frac{1}{2\pi i}\int_{\infty+it}^{-\infty+it} dz\ f(z)\ \frac{\beta}{2}\left(\cot\frac{\beta z}{2} + i\right)$$

— because $\cot\frac{\beta z}{2} - i$ decays rapidly in lower HP
$\cot\frac{\beta z}{2} + i$ decays rapidly in upper HP.

Exercise:
We find that, by completing contours in UHP and LHP respectively, provided that

$$-\beta < \tau < \beta,$$

$$G_E^{\beta}(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i\vec{k}\cdot\vec{x}} \frac{1}{1-e^{-\beta\omega_k}}\left[e^{-\omega_k|\tau|} + e^{\omega_k(|\tau|-\beta)}\right]$$

The expression

$$\frac{1}{1-e^{-\beta\omega_k}}\left[e^{-\omega_k|\tau|} + e^{\omega_k(|\tau|-\beta)}\right]$$

agrees with $G_+^{\beta} \sim e^{-i\omega_k t} + e^{-\beta\omega_k} e^{i\omega_k t}$

continued to $t = i\tau \quad \tau < 0$

$G_- \sim e^{i\omega_k t} + e^{-\beta\omega_k} e^{-i\omega_k t}$

continued to $t = i\tau \quad \tau > 0$

So -- the analytically continued cylinder Green function agrees with $G_+^\beta G_-^\beta$ in the strip, and therefore coincides with the function we constructed earlier.

Now, these observations can be generalized to 2 pt function on arbitrary static spacetime

Consider KG eqn

$$\left[\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}\,g^{\mu\nu}\partial_\nu + m^2\right]\phi(x) = 0$$

in the case of a static spacetime
— $g_{\mu\nu}$ independent of t
$g_{0i} = 0$

So $\left[g^{00}\frac{\partial^2}{\partial t^2} + \frac{1}{\sqrt{g}}\partial_i\sqrt{g}\,g^{ij}\partial_j + m^2\right]\phi(x) = 0$

or $\left[\frac{\partial^2}{\partial t^2} + g_{00}\left(\frac{1}{\sqrt{g}}\partial_i\sqrt{g}\,g^{ij}\partial_j + m^2\right)\right]\phi = 0$

Write this as $\left[\frac{\partial^2}{\partial t^2} + K\right]\phi = 0$

where K is a positive differential operator that acts on $\vec{x}$ dependence of $\phi(\vec{x}, t)$.
(Just $-\vec{\nabla}^2 + m^2$ for flat spacetime)

We can diagonalize K,

$$K u_i(\vec{x}) = \omega_i^2 u_i(\vec{x}),$$

and normalize modes so that

$$\int d^3x \sqrt{g^{00} h}\; u_i(\vec{x})^* u_j(\vec{x}) = \delta_{ij}$$

Then pos freq solutions normalized wrt KG inner product are

$$\frac{1}{\sqrt{2\omega_i}} u_i(\vec{x}) e^{-i\omega_i t}$$

cf page 2.7. Here $\sqrt{h}$ is induced volume element on a time slice, and $\sqrt{g^{00}}$ enters since $n^\mu \partial_\mu = \sqrt{g^{00}}\, \partial_t$.

Expanding fields in terms of these modes, we have

$$\phi(t,\vec{x}) = \sum_i \frac{1}{\sqrt{2\omega_i}} \left[u_i(\vec{x}) e^{-i\omega_i t} a_i + u_i(\vec{x})^* e^{i\omega_i t} a_i^\dagger \right],$$

Defining vacuum by $a_i |0\rangle = 0$, we have two point function

$$G_+(t,\vec{x},\vec{y}) = \langle 0| \phi(t,\vec{x})\, \phi(0,\vec{y}) |0\rangle$$

$$= \sum_i \frac{1}{2\omega_i} u_i(\vec{x}) u_i(\vec{y})^* e^{-i\omega_i t}$$

We can also evaluate expectation value in thermal ensemble

$$G_+^\beta(t,\vec{x},\vec{y}) = \langle \phi(t,\vec{x})\,\phi(0,\vec{y})\rangle_\beta$$

$$= \sum_i \frac{1}{2\omega_i} u_i(\vec{x})u_i(\vec{y})^* \Big[\langle a_i a_i^\dagger \rangle_\beta e^{-i\omega_i t} + \langle a_i^\dagger a_i \rangle_\beta e^{i\omega_i t} \Big]$$

(Here we've used $\sum_{\omega_i=\omega} u_i(\vec{x})u_i(\vec{y})^* = \sum_{\omega_i=\omega} u_i(\vec{x})^* u_i(\vec{y})$

– idea is that $u_i(\vec{x})e^{-i\omega_i t}$, $u_i(\vec{x})^* e^{-i\omega_i t}$

are both pos freq solns with the same frequency so we sum over both if $u_i(\vec{x})$ is not real.)

$$= \sum_i \frac{1}{2\omega_i} u_i(\vec{x})u_i(\vec{y})^* \frac{1}{1-e^{-\beta\omega_i}} \Big[e^{-i\omega_i t} + e^{-\beta\omega} e^{i\omega_i t} \Big]$$

Same arguments as before can be used to continue G_+ G_- G_+^β G_-^β away from real axis. We would like to show that these continued functions are given by Euclidean Green functions on the imaginary t axis.

The Euclidean Green function satisfies

$$g^{00}\Big[\frac{\partial^2}{\partial\tau^2} - K\Big] G_E(t,\vec{x},\vec{y}) = \frac{-1}{\sqrt{g}}\,\delta(t)\,\delta(\vec{x}-\vec{y})$$

or $$\Big[\frac{\partial^2}{\partial\tau^2} - K\Big] G_E(t,\vec{x},\vec{y}) = \frac{-1}{\sqrt{g^{00}h}}\,\delta(t)\,\delta(\vec{x}-\vec{y})$$

— δ^4 is normalized suitably for integration against the invariant volume element, and we used
$$g = g_{00} h \quad \text{where} \quad g_{00} = g^{00\,-1}.$$

The zero Temp Green function can be represented as
$$G_E(t, \vec{x}; \vec{y}) = \int \frac{dK^0}{2\pi} \sum_i u_i(\vec{x}) u_i(\vec{y})^* e^{ik^0\tau} \frac{1}{K_0^2 + \omega_i^2}$$

since completeness of solutions $\Rightarrow$
$$\sum_i u_i(\vec{x}) u_i(\vec{y})^* = \frac{1}{\sqrt{g^{00} h}} \delta(\vec{x} - \vec{y})$$

We can do the K^0 integral just as before to show that this agrees with G_+ and G_- extended to the imaginary axis.

At finite temperature, consider the Green function periodic in τ with period β. It is
$$G_E^\beta(t, \vec{x}, \vec{y}) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \sum_i u_i(\vec{x}) u_i(\vec{y})^* e^{\frac{2\pi i n \tau}{\beta}} \frac{1}{\left(\frac{2\pi n}{\beta}\right)^2 + \omega_i^2}$$

We can do the sum, as before, to show that G_E agrees with G_+^β and G_-^β continued to the imaginary axis in the strip
$$-\beta < \tau < \beta.$$

We have learned then, that on any static spacetime, correlation functions of free scalar fields on the spacetime, in the thermal ensemble, are obtained from the Euclidean Klein-Gordon Green function on the corresponding Euclidean "section" of the spacetime (imaginary time), by analytic continuation of the Green function to real time.

Now, how does this apply to quantum field theory on Rindler spacetime? Recall the relation between Rindler coordinates in the wedge R and Minkowski coordinates

$$z = \xi \cosh \eta$$
$$t = \xi \sinh \eta$$

Now continue to imaginary Rindler time

$$\eta = i\eta_E$$

thus

$$z = \xi \cos \eta_E$$
$$\tau = \xi \sin \eta_E, \text{ where } t = i\tau$$

The coordinates z and τ are periodic functions of (Euclideanized) Rindler time with period 2π.

We can now understand why the Minkowski vacuum looks like a thermal state to a Rindler observer. Because

$\langle 0, Min | \phi(t, \vec{x}) \phi(0) | 0, Min \rangle$

is the analytic continuation of the Euclidean KG Green function $G_E(\tau, \vec{x})$ to real time. Regarded as a function of the Euclidean Rindler coordinates $G_E(\tau, \vec{x}) = G_E(\eta_E, \xi, x, y)$ satisfies the Rindler K-G equation

$$\left[\xi^{-2}\partial_{\eta_E}^2 + \frac{1}{\xi}\partial_\xi \xi \partial_\xi + \partial_x^2 + \partial_y^2\right] G_E(\eta_E, \xi, x, y; \xi', x', y')$$

$$= \frac{1}{\xi}\delta(\eta_E)\delta(\xi-\xi')\delta(x-x')\delta(y-y')$$

and is periodic in η_E with period 2π. Hence, its continuation to real Rindler time η is the Rindler thermal state with $\beta = 2\pi$

A peculiar feature of the above argument is that although the wedge R has a boundary that can be crossed by geodesics, the continuation giving

$$z = \xi \cos\eta_E$$
$$\tau = \xi \sin\eta_E$$

covers the whole Euclidean z, τ plane. The horizon at $\eta = 0$ becomes the coordinate singularity at the origin of a polar coordinate system.

Remark: More detail about the analytic properties of thermal Green functions can be found in:

S.A. Fulling and S.N.M. Ruijsenaars "Temperature, Periodicity, and Horizons" Phys. Rep. 152 (1987) 135.

One reason that the above argument is significant is that we may hope to generalize it to interacting fields on Rindler spacetime. We have already noted that the KMS condition is quite general. Hence

$$\langle \phi(t-i\beta, \vec{x})\, \phi(0, \vec{y}) \rangle_\beta$$
$$= \langle \phi(0, \vec{y})\, \phi(t, \vec{x}) \rangle_\beta$$

In addition, since fields commute at spacelike separation

$$\langle \phi(t, \vec{x})\, \phi(0, \vec{y}) \rangle_\beta \text{ and } \langle \phi(0, \vec{y})\, \phi(t, \vec{x}) \rangle_\beta$$

can be analytically continued one to the other; Together with the KMS condition, this tells us that

$$\langle \phi(t, \vec{x})\, \phi(0, \vec{y}) \rangle$$

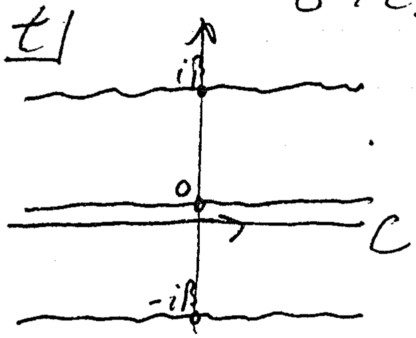
is the boundary value of a single function analytic in the complex t plane, with period $i\beta$.

Now suppose we try to work backwards. Suppose we know that

$$\langle \phi(t,\vec{x})\, \phi(0,\vec{y}) \rangle$$

is the boundary value of a function $G(t,\vec{x},\vec{y})$ that is analytic in the cut t plane, with cuts only at $\mathrm{Im}\, t = \beta n$, and is periodic in t with period $i\beta$. Suppose also that the discontinuity of G across the cut on the real axis is such that

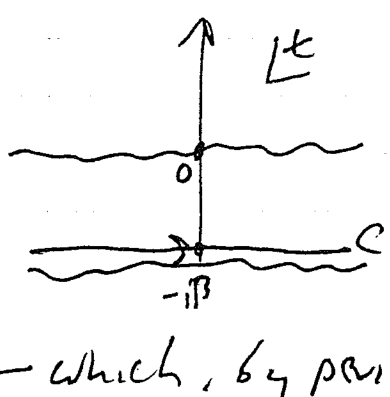
$$G(t,\vec{x},\vec{x})_{\substack{\text{above}\\ \text{cut}}} = G(-t,\vec{x},\vec{x})_{\substack{\text{below}\\ \text{cut}}}$$

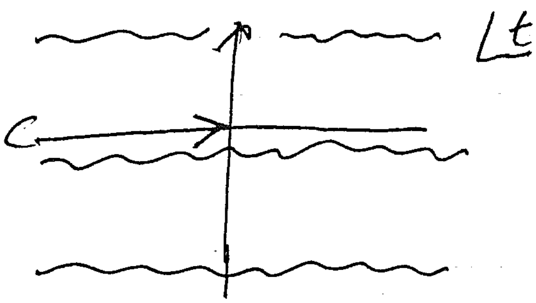


The response of a static detector to the field fluctuations is given by a response function

$$\tilde{G}(\omega,\vec{x}) = \int dt\, e^{-i\omega t}\, G(t,\vec{x},\vec{x})_{\substack{\text{below}\\ \text{cut}}}$$

(Cf page 3.8). Now -- for $\omega > 0$, we can distort the contour to the bottom of the strip

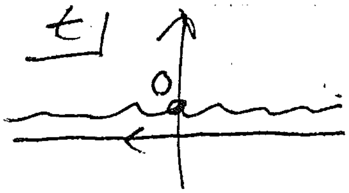




— which, by periodicity, is same as contour above cut on real axis.

We can put the contour back below the cut by replacing $G(t) \to G(-t)$.
So

$$\tilde{G}(\omega, \vec{x}) = \int dt\, e^{-i\omega(t-i\beta)}\, G(-t, \vec{x}, \vec{x})_{\text{below cut}}$$



$$= e^{-\beta\omega} \int dt\, e^{i\omega t}\, G(t, \vec{x}, \vec{x})_{\text{below cut}}$$

(replacing $t \to -t$)

$$\tilde{G}(\omega, \vec{x}) = e^{-\beta\omega}\, \tilde{G}(-\omega, \vec{x}) \qquad (\omega > 0)$$

Thus — positive frequency response is suppressed relative to negative frequency response by a Boltzmann factor, $e^{-\beta\omega}$. A detector coupled to frequency ω mode of the field will be thermally occupied when in equilibrium with the field fluctuations.

All of the assumed properties of $G(t, \vec{x}, \vec{x})$ should be satisfied by interacting quantum fields in the vacuum state, when G is expressed in terms of Rindler coordinates. Hence the vacuum looks to the uniformly accelerated observer like a thermal state with $T = a/2\pi$, even if the fields are <u>not</u> free.

(This idea of sliding down the contour was used by J. Hartle and S. Hawking, Phys. Rev D<u>13</u> (1976) 2188, in a related context.)