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4. Black Hole Radiance

Schwarzschild geometry:

In 1916 K. Schwarzschild found that the empty space Einstein equation

$$R_{\mu\nu} = 0$$

admits the spherically symmetric solution

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{1}{1-\frac{2M}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

(here $G = 1$; otherwise replace $M \to GM$)
We may define M as the total mass of gravitating body at the origin; this then agrees with gravitational mass M of Newtonian theory for large r.

It was later shown by G. Birkhoff (1923) that above is the unique spherically symmetric solution of $R_{\mu\nu} = 0$. Thus, in particular, spherical symmetry implies geometry is static. ("Uniqueness" meaning, of course, up to reparametrization.)
"Birkhoff's theorem" says that a pulsating spherical body cannot radiate gravitational waves. (The graviton is spin-2. Cf: No $0 \to 0$ atomic transitions accompanied by emission of spin-1 photon.)

According to Birkhoff's theorem, the Schwarzschild metric describes geometry exterior to a spherical star, even as the star undergoes (spherically symmetric) gravitational collapse. When the surface of the star collapses inside the (coordinate) singularity of the metric at $r = 2M$, we say that a (static) black hole has formed.

To understand this geometry better, consider the radial null geodesics. For this purpose, it is convenient to make a coordinate transformation

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left(dt^2 - \left[\frac{dr}{1 - \frac{2M}{r}}\right]^2\right) - r^2 d\Omega^2$$

Let $\quad dr_* = \dfrac{r\,dr}{r - 2M} \Rightarrow$

$$r_* = \int \left(1 + \frac{2M}{r - 2M}\right) dr = r + 2M \ln\left(\frac{r - 2M}{2M}\right)$$

arbitrary constant of integration

We have

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left(dt^2 - dr_*^2\right) - r^2 d\Omega^2$$

So radial null geodesics at $r_* = \pm t + \text{const}$

The "tortoise coordinate" goes to $r_* = -\infty$ as the sphere $r = 2M$ is asymptotically approached — it is the radial coordinate apportioned according to how much time (as perceived by a "FIDO" at fixed position in Schwarzschild coordinate) an infalling photon spends as it approaches $r=2M$. To this FIDO, the photon does not reach $r=2M$ until $t=+\infty$; and similarly a photon moving radially out emerged from the $r=2M$ surface at $t=-\infty$.

This happens because of the very strong gravitational field that causes the gravitational time delay

$$dt = \frac{d\tau}{\sqrt{1-\frac{2M}{r}}}$$

of a static clock to diverge at $r=2M$. Evidently, the surface $r=2M$ is an event horizon to FIDO at fixed Schwarzschild position, for a signal he sends to the horizon does not reach it until $t=\infty$, and a signal received from it emerged at $t=-\infty$. This is very analogous to Rindler spacetime. Let's pursue this analogy further.

Recall Rindler metric (suppressing $-dx^2-dy^2$)

$$ds^2 = \xi^2\left(d\eta^2 - d\xi_*^2\right) \qquad \xi_* = \ln \xi$$

The coordinate singularity at $\xi=0$ is an artifact of the motion of Rindler observers, and is not seen by freely falling observers. Consider the affine parameter of a photon, to appreciate that nothing special happens to geodesics at horizon. This parameter λ may be defined by $d\lambda = dt/\omega$ where dt is time as measured by observer who perceives frequency of photon to be ω. (Then $d\lambda$ is frame independent, although dt and ω are not; cf. proper time along path, which is frame independent parameter for massive freely falling object.)

For geodesics in static spacetime, there is a conservation law

$$K_0 = g_{00} K^0 \equiv E = \text{const}$$

For a null geodesic with $x, y = $ constant in Rindler spacetime

$$\xi^2 \frac{d\eta}{d\lambda} = E \quad \text{and} \quad 0 = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$$

$$= \xi^2 \left[\left(\frac{d\eta}{d\lambda}\right)^2 - \left(\frac{d\xi_*}{d\lambda}\right)^2 \right]$$

It is convenient to introduce null coordinates

$$\begin{aligned} u &= t - \xi_* \\ v &= t + \xi_* \end{aligned} \Rightarrow ds^2 = e^{v-u}\, du\, dv$$

and the affine parameter λ satisfies

$$e^{v-u} \frac{1}{2} \frac{d}{d\lambda}(u+v) = E = \text{constant}$$

From $\frac{du}{d\lambda}\frac{dv}{d\lambda} = 0$ we see that either u = const or v = const (geodesics moving up or down)

For u = const, we have

$$d\lambda = \frac{1}{2E} e^{-u} \int e^{v} dv$$

$$\text{or } \lambda = \text{const} + \frac{1}{2E} e^{v-u}$$

$$\lambda \sim c e^{v}$$

For v = const, we have

$$d\lambda = \frac{1}{2E} e^{v} \int e^{-u} du$$

$$= \text{const} - \frac{1}{2E} e^{v-u}$$

$$\lambda \sim -c' e^{-u}$$

The photons reach horizon $u = -\infty$, $v = \infty$ at finite value of affine parameter λ.

It is natural to use affine parameters of null geodesics as new coordinates, and so define

$$V = e^{v}$$

$$U = -e^{-u}$$

$$\Rightarrow ds^2 = dU\, dV$$

this is just the flat Minkowski metric

$$ds^2 = dt^2 - dz^2$$

— which is easily extended beyond the Rindler horizon.

Let's apply this lesson to the Schwarzschild geometry

$$ds^2 = \left(1 - \frac{2M}{r}\right)(dt^2 - dr_*^2) - r^2 d\Omega^2$$

For radial null geodesics we have

$$k_0 = g_{00} k^0 = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} = \text{const}$$

Introduce null coordinates

$$u = t - r_*$$
$$v = t + r_*$$

$$e^{(v-u)/4M} = e^{r_*/2M} = \left(\frac{r}{2M} - 1\right)e^{r/2M}$$

$$\Rightarrow \quad ds^2 = \underbrace{\frac{2M}{r} e^{-r/2M}} e^{(v-u)/4M}\, du\, dv \qquad (\text{suppress } -r^2 d\Omega^2)$$

near the horizon, $r \simeq 2M$, we have precisely the Rindler metric, up to a constant

Thus, we transform as before

$$\left.\begin{aligned} U &= -e^{-u/4M} \\ V &= e^{v/4M} \end{aligned}\right\} \text{ these are } \underline{\text{affine}} \text{ parameters of radial null geodesics } \underline{\text{at horizon}}$$

Since $dU\,dV = \frac{1}{16M^2} e^{v-u}\,du\,dv$,

we have
$$ds^2 = \frac{32M^3}{r} e^{-r/2M}\,dU\,dV - r^2 d\Omega^2$$

These are the Kruskal coordinates for Schwarzschild geometry, and they readily admit the <u>Kruskal extension</u> past the past and future horizons.

But the singularity at $r=0$ remains; it is a genuine singularity in the curvature tensor, and is an intrinsic boundary of the spacetime beyond which geodesics cannot be extended

In the region of Schwarzschild exterior to horizon, $r > 2M$, we have
$$e^{r_*/4M} = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{r/4M}$$

and so our coordinate transformation may be expressed
$$U = -\left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \exp\left[(r-t)/4M\right]$$
$$V = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \exp\left[(r+t)/4M\right]$$

$$U < 0$$
$$V > 0$$

These coordinates may be smoothly extended into other quadrants of UV plane

Note that it is rather natural to adopt $4M$ as a unit of length, in which case

$$ds^2 = \frac{1}{2r} e^{-2r}\, dU dV - r^2 d\Omega^2$$

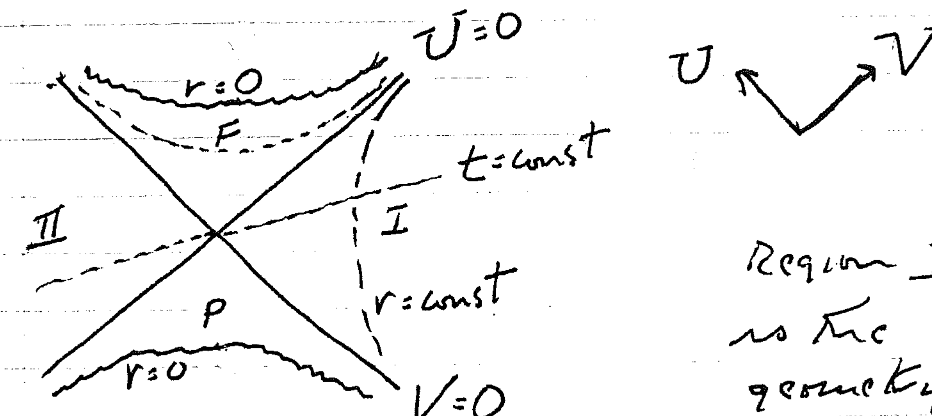
looks a little nicer. We may also, if we wish, introduce X, T by

$$U = T - X$$
$$V = T + X$$

so that $ds^2 = \frac{1}{2r} e^{-2r} (dT^2 - dX^2) - r^2 d\Omega^2$,

and radial null geodesics are $45°$ angles in T, X plane. If t, r are analogous to Rindler η, ξ, then T, X are analogous to Minkowski t, z

The causal structure of the spacetime may be summarized by a "Kruskal diagram" of the T, X plane. Each point of the diagram represents a 2 sphere, or, we may think of the diagram as representing a $\theta, \phi =$ constant slice through the spacetime.



Region I of the diagram is the Schwarzschild geometry exterior to horizon

In this region $U/V = -e^{-t/2M}$,
so surfaces of constant Schwarzschild time t are straight lines through the origin,
and
$$UV = -\left(\frac{r}{2M} - 1\right) e^{r/2M},$$
so surfaces of constant Schwarzschild radius r are hyperbolas

Now we can extend these lines and hyperbolas into the other quadrants of the U, V plane, as we analytically extended the Rindler metric.

Region F

$$U = \left(1 - \frac{r}{2M}\right)^{\frac{1}{2}} \exp\left[(r+t)/4M\right] \qquad U > 0$$
$$V = \left(1 - \frac{r}{2M}\right)^{\frac{1}{2}} \exp\left[(r-t)/4M\right] \qquad V > 0$$

Region II

$$U = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \exp\left[(r-t)/4M\right] \qquad U > 0$$
$$V = -\left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \exp\left[(r+t)/4M\right] \qquad V < 0$$

Region P

$$U = -\left(1 - \frac{r}{2M}\right)^{\frac{1}{2}} \exp\left[(r+t)/4M\right] \qquad U < 0$$
$$V = -\left(1 - \frac{r}{2M}\right)^{\frac{1}{2}} \exp\left[(r-t)/4M\right] \qquad V < 0$$

The logic of this extension is as for Rindler spacetime. Since geodesics move up through past horizon at $t=-\infty$ and down through future horizon at $t=+\infty$, we need a region "before" $t=-\infty$ and "after" $t=+\infty$ to extend these geodesics. And since there are also upcoming geodesics in Region F, downgoing geodesics in region P, we learn of the need for region II to continue these geodesics, even though I and II are causally disconnected.

The new feature, not found in Rindler spacetime, is the singularity at $r=0$, or in Kruskal coordinates, at

$$UV = 1 \quad \text{regions F and P}$$

– this is as far as our analytic extension of Schwarzschild geometry can take us. An observer who falls past the future event horizon inevitably meets this singularity (Note that $r=0$ is a spacelike surface, for r is a spacelike coordinate inside horizon.) Similarly, an observer who enters region I from region P must have come from the past singularity that bounds that region.

Because of the singularity, the horizon has a more fundamental significance for the Schwarzschild geometry than for Rindler. All observers agree, regardless of their state of motion, that once the future horizon is passed, it is impossible to escape again to $r=\infty$ and furthermore, that the dreaded singularity will inevitably be met, bringing trajectory to an end in region of ∞ tidal force, in finite amount of proper time.

The constant r hyperbolas are invariant under boosts in the T-X plane. This is because

$$UV = T^2 - X^2 = -\left(\frac{r}{2M} - 1\right) e^{r/2M}, \text{ Region I}$$

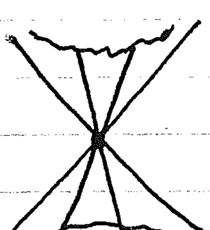
etc in other regions

defines r as a function of $T^2 - X^2$, which is invariant under boosts. All the points on the hyperbola have the same proper acceleration relative to local inertial frame, since proper acceleration is frame independent. All of the constant t surfaces in region I are related by boosts, as

$$\frac{T}{X} = \frac{\frac{1}{2}(U+V)}{\frac{1}{2}(V-U)} = \tanh t/4M$$

and a boost in the T-X plane takes

$$\tanh(t/4M) \to \tanh(t/4M + \theta).$$



Similarly, the constant t lines that connect the past singularity in region B with the future singularity in region F are related by Lorentz boosts (and so all have the same proper length) This length (really a proper time) is

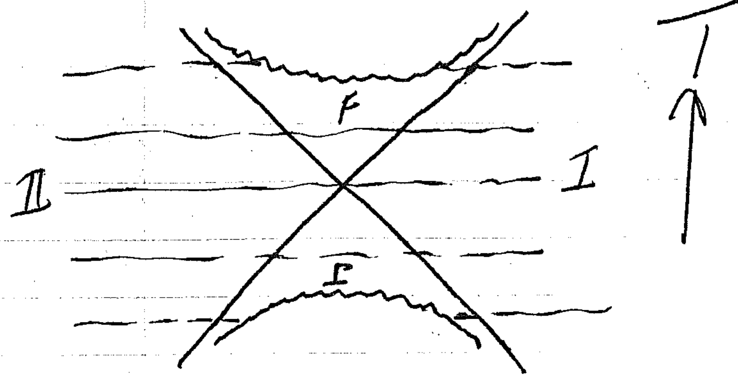
$$2 \int_0^{2M} \frac{dr}{\sqrt{\frac{2M}{r}-1}} = 4M \int_0^1 dx \sqrt{\frac{x}{1-x}}$$

Let $\begin{array}{l} x = \sin^2\theta \\ dx = 2\sin\theta\cos\theta d\theta \end{array} \Rightarrow = 4M\int_0^{\pi/2} d\theta \, 2\sin\theta\cos\theta \frac{\sin\theta}{\cos\theta} = 2\pi M$

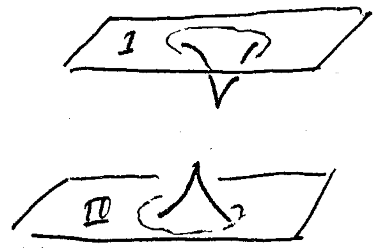
— πM is the longest possible proper time for fall from the horizon to the singularity.

It may seem somewhat surprising at first that extended Schwarzschild geometry has two spacelike singularities. Geodesics that pass through past horizon cannot be extended indefinitely back in time, just as those that pass through future horizon cannot be extended indefinitely forward in time. To better appreciate the structure of spacetime, note first of all that, while the Schwarzschild geometry exterior to horizon is static — $\frac{\partial}{\partial t}$ is timelike Killing vector — the interior region is not static, as $\frac{\partial}{\partial t}$ becomes spacelike there. Thus, we may regard the geometry as dynamical, at least inside the horizon. The way we describe this dynamics depends on how we "foliate" the spacetime with spacelike surfaces.

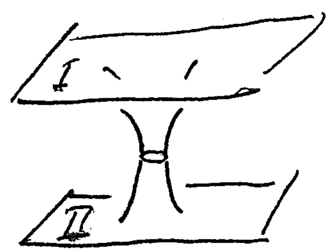
E.g. suppose we choose surfaces of constant T, as shown



Then, the "initial" spacetime for T sufficiently "early", consists of two disconnected asymptotically flat components, each containing a singularity (cloaked by past horizon).



Then, at some time, the two singularities join together and smooth out forming a neck or "wormhole" that connects the two components.

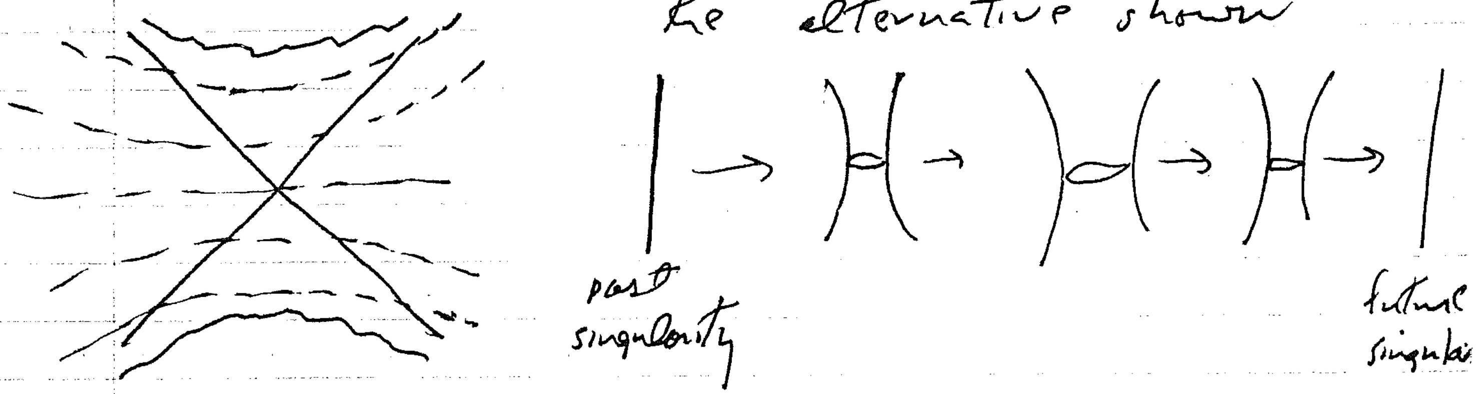


The neck widens, reaching a maximal proper radius $r = 2M$, at which point the neck is instantaneously static, and the event horizons of regions I and II instantaneously join.

Then the neck recontracts, eventually pinching off as the two singularities reappear, and the spacetimes disconnect.

As is clear from the Kruskal diagram, the process of wormhole formation and recollapse occurs so rapidly that it is impossible to traverse the wormhole and travel from I to II. Whoever dares to try inevitably encounters the dreaded singularity.

Just to emphasize that the picture of dynamics depends on the foliation, consider the alternative shown



Here the spacetime comes into being as a spacelike past singularity, flares out, reaches a maximal girth and recollapses to a spacelike future singularity.

In representing causal structure of a spacetime, it is often convenient to use coordinates that parametrize spatial and temporal ∞ by finite values of coordinates.

Example: flat spacetime

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2$$

Change coordinates

$$t - r \equiv u = \tan u'/2$$

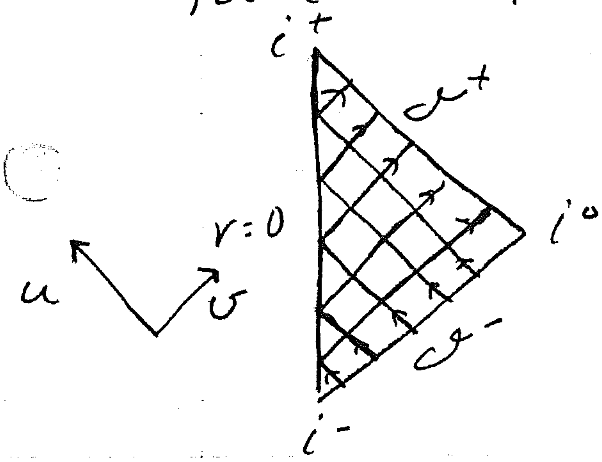
$$t + r \equiv v = \tan v'/2$$

$$\Rightarrow \quad dt^2 - dr^2 = du\,dv = \tfrac{1}{4}\sec^2\!\left(\tfrac{u'}{2}\right)\sec^2\!\left(\tfrac{v'}{2}\right) du'\,dv'$$

Radial null geodesies at $u = \text{const}$ or $v = \text{const}$, and hence $u' = \text{const}$ or $v' = \text{const}$

Of course, since $u = \pm\infty$ is now at $u' = \pm\pi$
$v = \pm\infty$ is at $v' = \pm\pi$
– the metric must blow up at $u', v' = \pm\pi$, so that proper distances remain ∞.

(If we wish, we can conformally rescale metric
$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$$
so that ∞ is brought to a finite distance from origin; this transformation preserves null geodesics)



A diagram ("conformal diagram" or "Penrose diagram") of the spacetime is shown, with radial null geodesics drawn.

Regions at ∞ are

i^+: $u = \infty$, $v = \infty$. This is future timelike ∞ that is $t \to \infty$ faster than r. (All timelike paths arrive at i^+ in the future.)

i^-: $u = -\infty$, $v = -\infty$. This is past timelike ∞. that is $t \to -\infty$ faster than $-r$ (All timelike paths originate at i^-)

i^0: $u = -\infty$, $v = \infty$. This is spacelike ∞ $r \to \infty$ faster than t (All constant time slices contain the "point" i^0)

$\mathcal{I}^+$: $v = \infty$, u = finite. This is future null infinity, $r \sim t + \text{const}$. Outgoing null geodesics reach $\mathcal{I}^+$

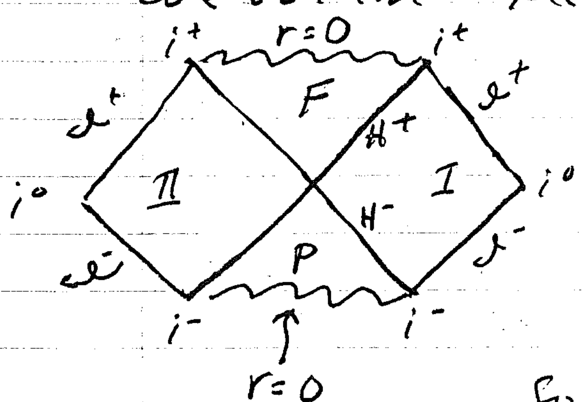
$\mathcal{I}^-$: $u = -\infty$, v = finite. This is past null infinity, $r \sim -t + \text{const}$. Incoming null geodesics originate on $\mathcal{I}^-$

$r=0$: $u = v$ – this is where a radial geodesic "crosses over" from incoming to outgoing

We can apply the same transformation to Kruskal coordinates of the extended Schwarzschild geometry,

$$U = \tan(U'/2),$$
$$V = \tan(V'/2).$$

We obtain the diagram:



Recall: Horizons at $U=0$, $V=0$

$$U = -e^{-u/4M}$$
$$-V = e^{v/4M}$$ Region I

where $u = t - r_*$, $v = t + r_*$

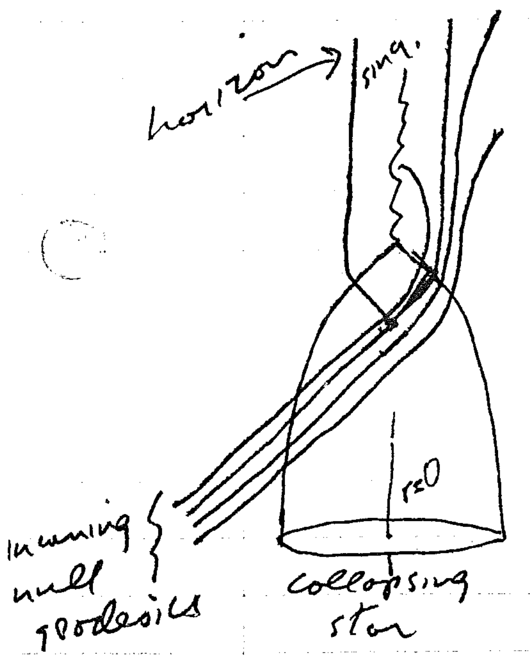
So $UV = -e^{r_*/2M}$
$U/V = -e^{-t/2M}$

Thus, e.g.
- static trajectory contains $-UV$ = finite, $U/V = 0$ or $\begin{matrix} U=0 \\ V=\infty \end{matrix}$
- Time slices contain $-UV = \infty$, U/V = finite or $\begin{matrix} U=-\infty \\ V=\infty \end{matrix}$

etc.

Spherically Symmetric Collapse:

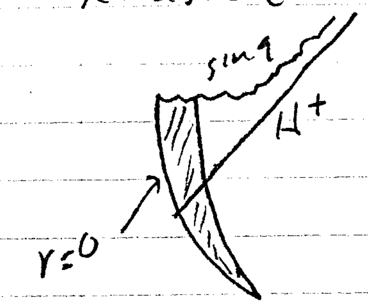
The Kruskal metric describes an "eternal" static black hole such that singularity is present in asymptotic past. It does not describe the "realistic" case of a BH that forms from a collapsing star. For the geometry of a collapsing star has no past horizon or past singularity. (Regions P and II are not part of this geometry)



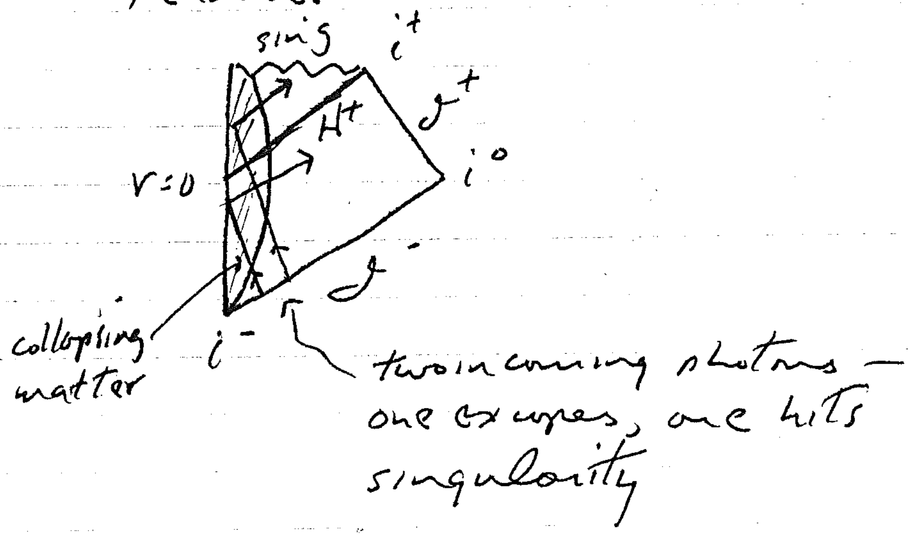
Although there are strictly speaking no outgoing null geodesics that punch through the horizon at $t=-\infty$ (there is no horizon at $t=-\infty$), there are null geodesics that pass through the center of collapsing star, just before the formation of the horizon, and are delayed arbitrarily long before finally escaping. (Assume here that "photons" interact only with the geometry, not with the collapsing matter.) The photon that arrives at the center simultaneously with the formation of the horizon remains frozen in the future at $r=2M$, unable to pull away.

Radial null geodesics are most easily kept track of on corresponding Penrose or Kruskal diagram

Kruskal:



Penrose:



It is a crucial property of the spacetime of the collapsing star that some incoming photons escape only after an arbitrarily long time has elapsed. Furthermore, when they finally emerge, these photons are very strongly redshifted, relative to their incoming frequency. This observation will allow us to investigate the radiation from a collapsed star by doing QFT in region I of the Kruskal spacetime, with appropriate <u>boundary conditions</u> on the past horizon.

The modes of the field that are delayed very long by the newly formed horizon, and are very strongly redshifted, play a central role in black hole radiance, as we shall see.

Two "easy" derivations of black hole radiance:

- A FIDO close to the horizon of a Schwarzschild black hole has a large proper acceleration, relative to inertial observer.

Compute acceleration by considering radial geodesics

$$p_0/m = \tilde{E} = \text{const} \quad \text{and} \quad g_{\alpha\beta}p^\alpha p^\beta - m^2 = 0$$

$$\Rightarrow \quad \frac{1}{1-\frac{2M}{r}}\tilde{E}^2 - \frac{1}{1-\frac{2M}{r}}\left(\frac{dr}{d\tau}\right)^2 - 1 = 0$$

$$\text{or} \quad \left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1-\frac{2M}{r}\right) \Rightarrow \frac{dr}{d\tau} = \left[\tilde{E}^2 - \left(1-\frac{2M}{r}\right)\right]^{\frac{1}{2}}$$

$$\Rightarrow \quad \frac{d^2r}{d\tau^2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^{-1}\left(\frac{dr}{d\tau}\right)\frac{-2M}{r^2} = -\frac{M}{r^2}$$

So - acceleration of freely falling observer (FFO) relative to FIDO is

$$\frac{d^2s}{d\tau^2} = \left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}\frac{d^2r}{d\tau^2} = -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}$$

$$a_{proper} = \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}$$

$$\sim \frac{1}{4M}\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} \quad \text{for } r \sim 2M$$

To FFO, nothing special is happening at the horizon, and fluctuations of quantum fields should resemble those in Minkowski bath. But then FIDO should see Rindler bath at Temperature

$$T(r) = \frac{1}{2\pi}a_{proper}(r) \sim \frac{1}{8\pi M}\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}$$

When this radiation propagates to $r = \infty$, it is red shifted by the factor

$$\left[\frac{g_{00}(r)}{g_{00}(\infty)}\right]^{\frac{1}{2}} = \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}$$

Thus, a distant observer sees

$$\boxed{T(\infty) = \frac{1}{8\pi M}}$$

- The transformation from Schwarzschild to Kruskal coordinates, in all four quadrants, is <u>periodic</u> in Schwarzschild time t with period $8\pi M i$

e.g. $$U = -\left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{(r-t)/4M}$$
$$V = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{(r+t)/4M} \quad \text{(Region I)}$$

Now consider a "state" such that the two point correlation function $\langle \phi(x)\, \phi(y) \rangle$ is analytic on the "Euclidean section" (continuation to imaginary time) of Schwarzschild (Kruskal) geometry, and matches the Euclidean Kruskal Klein-Gordon Green function $G_E(x, y)$ there.

(Note: Euclidean Schwarzschild has no singularity, as we'll discuss later) this Kruskal Green function is also a Schwarzschild Green function, but periodic in $\tau = it$ with period $\beta = 8\pi M$. Thus field correlations measured by a FIDO are thermal, with

$$T = \frac{1}{8\pi M}$$

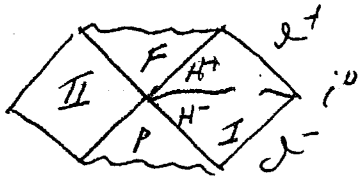
— this is temperature in units of Schwarzschild time frequency, and corresponds to a locally measure temperature

$$T(r) = \frac{1}{8\pi M}\left[g_{00}(r)\right]^{-\frac{1}{2}}$$

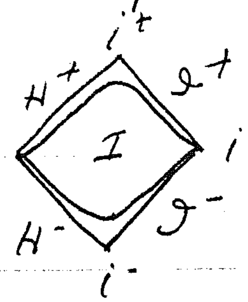
Remark: Both arguments suggest a black hole in a thermal bath, rather than a black hole emitting radiation into surrounding empty space. To better appreciate what this "state" is, to see how to describe a BH radiating into empty space, and to gain a firmer grasp of what is going on, we now proceed to more detailed study of QFT in the BH background.

QFT in Kruskal Geometry

To describe the "realistic" case of a BH that forms by gravitational collapse, we will actually need to do QFT on the background of a collapsing star. But first we'll consider the case of an "eternal" black hole — the extended Kruskal geometry, with both past and future horizons.



Note that region I of the Kruskal extension is itself a globally hyperbolic spacetime (though not geodesically complete) — it has Cauchy surfaces. (Cf. the Rindler wedge.)



Imagine distorting the Cauchy surface in I, so that it lies close to $H^- \cup \mathcal{I}^-$ or $H^+ \cup \mathcal{I}^+$

In fact, in the massless case, it is legitimate to regard $H^- \cup \mathcal{I}^-$ or $H^+ \cup \mathcal{I}^+$ to be a Cauchy surface for the equation

$$\nabla_\mu \nabla^\mu \phi = 0$$

Any wave packet, when propagated backward in time, eventually crosses H^- or $\mathcal{I}^-$; and when propagated forward in time crosses H^+ or $\mathcal{I}^+$. (For $m^2 \neq 0$, we also need to specify data on i^-, i^+ respectively.) We'll mainly consider the massless case, because of this simplifying feature that $\mathcal{H}^- \cup \mathcal{I}^-$ or $\mathcal{H}^+ \cup \mathcal{I}^+$ may be regarded as Cauchy.

The Klein-Gordon eqn

$$\left[\frac{1}{\sqrt{g}}\partial_\mu \sqrt{g}\, g^{\mu\nu}\partial_\nu + m^2\right] u(x) = 0$$

has solutions that can be expanded in terms of spherical harmonics, because of the spherical symmetry of Schwarzschild geometry.
Write

$$u = f_\ell(r_*, t)\,\frac{1}{r}\,Y_{\ell m}(\theta, \phi),$$

and find (exercise)

$$(\nabla_\mu \nabla^\mu + m^2)\, u = 0 \Rightarrow$$

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell(r_*)\right] f_\ell(r_*, t) = 0$$

where the "effective potential" is

$$V_\ell(r_*) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} + m^2\right]$$

Recall $r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right)$, so

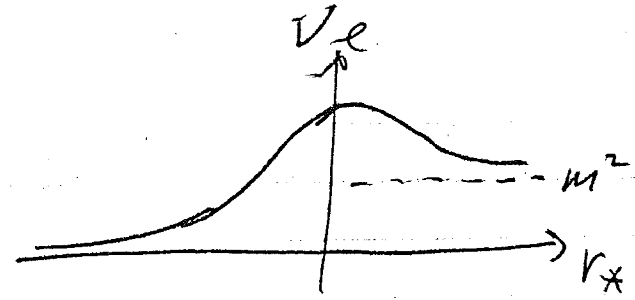
$$r_* \sim 2M \ln\left(\frac{r}{2M} - 1\right) \Rightarrow \frac{r}{2M}\left(1 - \frac{2M}{r}\right) \sim e^{r_*/2M}$$

$$\text{or } \left(1 - \frac{2M}{r}\right) \sim e^{r_*/2M} \text{ for } r \sim 2M$$

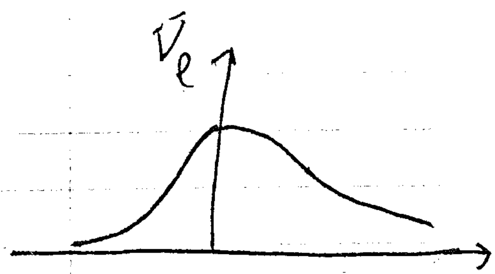
Hence $V_\ell(r_*) \sim e^{r_*/2M}\,(\;-\;)$ for $r_* \to -\infty$

$$V_\ell(r_*) \sim m^2\left(1 - \frac{2M}{r}\right) + \frac{\ell(\ell+1)}{r^2} + \cdots$$

for $r_* \to \infty$



Note that the potential rapidly turns off as the horizon is approached. This is because a wave approaching the horizon is very strongly blue shifted; when wave length is very short, and frequency very high, the mass, centrifugal barrier, and curvature perturb the propagating wave very little, and it obeys the (1+1)-dimensional wave equation in r_*, t coordinates



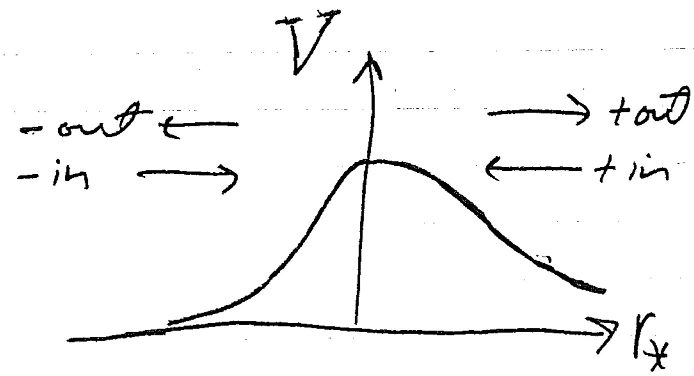
In the massless case $V_\ell \to 0$ for both $r_* \to -\infty$ and $r_* \to \infty$.

Consider now the waves of definite Schwarzschild frequency

$$f(r_*, t) = e^{\pm i\omega t} f_{\ell,\omega}(r_*)$$

$$\Rightarrow \quad \left[-\frac{\partial^2}{\partial r_*^2} - \omega^2 + V_\ell(r_*)\right] f_{\ell,\omega}(r_*) = 0$$

This has just the form of a 1-dimensional Schrodinger equation. By solving it, we can relate waves that are incoming from $r_* = -\infty$ (horizon) or $r_* = \infty$ to outgoing waves that are transmitted or reflected and reach $r_* = -\infty$ or $r_* = \infty$



In the $r_* \to -\infty$ region, solution has the general form

$$f(r_*) = A_{in}^{(-)} e^{i\omega r_*} + A_{out}^{(-)} e^{-i\omega r_*} \quad (r_* \to -\infty)$$

and for $r \to \infty$

$$f(r_*) = A_{out}^{(+)} e^{i\omega r_*} + A_{in}^{(+)} e^{-i\omega r_*} \quad (r_* \to +\infty)$$

Since the equation is linear, the $r_* \to +\infty$ solution depends linearly on $r_* \to -\infty$ solution. We may define an "S-matrix" for each partial wave l

$$\begin{pmatrix} A_{out}^{(+)} \\ A_{out}^{(-)} \end{pmatrix} = S \begin{pmatrix} A_{in}^{(-)} \\ A_{in}^{(+)} \end{pmatrix}$$

(defined so $S = 1$ for $V = 0$)

Because the "probability flux" is

conserved, $$\frac{d}{dr_*}\left(f^* \frac{d}{dr_*} f - \left(\frac{d}{dr_*} f^*\right) f \right) = 0 ,$$

we see that
$$|A_{in}^{(-)}|^2 - |A_{out}^{(-)}|^2 = |A_{out}^{(+)}|^2 - |A_{in}^{(+)}|^2$$
or $$|A_{in}^{(-)}|^2 + |A_{in}^{(+)}|^2 = |A_{out}^{(+)}|^2 + |A_{out}^{(-)}|^2$$

Thus, S is a unitary matrix.

A further property of S follows from the "time-reversal invariance" of the KG equation. If $f(r_*)$ solves the above eqn, then so does $f(r_*)^*$ — so we still have a solution after the replacements
$$A_{in}^{(-)} \to A_{out}^{(-)*}$$
$$A_{out}^{(-)} \to A_{in}^{(-)*}$$
$$A_{out}^{(+)} \to A_{in}^{(+)*}$$
$$A_{in}^{(+)} \to A_{out}^{(+)*}$$

Therefore $$\begin{pmatrix} A_{in}^{(+)*} \\ A_{in}^{(-)*} \end{pmatrix} = S \begin{pmatrix} A_{out}^{(-)*} \\ A_{out}^{(+)*} \end{pmatrix}$$

or $$\begin{pmatrix} A_{out}^{(-)} \\ A_{out}^{(+)} \end{pmatrix} = (S^*)^{-1} \begin{pmatrix} A_{in}^{(+)} \\ A_{in}^{(-)} \end{pmatrix}$$

Since $S^{-1} = S^{\dagger}$, this is $S^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

or $$\begin{pmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{pmatrix} = \begin{pmatrix} S_{22} & S_{21} \\ S_{12} & S_{11} \end{pmatrix} \Rightarrow S_{11} = S_{22}$$

The point of this analysis is that unitarity and time-reversal invariance allow us to relate the transmission and reflection of waves traveling toward and away from horizon.

We may define transmission and reflection coefficients by

$$\begin{matrix} A_{in}^{(-)} = 1 \\ A_{in}^{(+)} = 0 \end{matrix} \Rightarrow \begin{matrix} A_{out}^{(+)} = t \\ A_{out}^{(-)} = r \end{matrix}$$

Then $$\begin{pmatrix} t \\ r \end{pmatrix} = S \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{matrix} S_{11} = t \\ S_{21} = r \end{matrix}$$

So time reversal $\Rightarrow$ $S_{22} = t$.

or $S = \begin{pmatrix} t & S_{12} \\ r & t \end{pmatrix}$; unitarity $\Rightarrow S_{12} = -\frac{t}{t^*} r^*$

$$\Rightarrow S = \begin{pmatrix} t & -\left(\frac{t}{t^*}\right) r^* \\ r & t \end{pmatrix}$$

Hence

$$\begin{matrix} A_{in}^{(+)} = 1 \\ A_{in}^{(-)} = 0 \end{matrix} \Rightarrow \begin{pmatrix} A_{out}^{(+)} \\ A_{out}^{(-)} \end{pmatrix} = S \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} S_{12} \\ S_{22} \end{pmatrix}$$

$$= \left(\frac{t}{t^*}\right) \begin{pmatrix} -r^* \\ t^* \end{pmatrix}$$

We can change the phase convention for the incoming wave so that $A_{in}^{(+)} = (t^*/t)$, and then reflected and transmitted waves are given by

$$A_{out}^{(-)} = t^*$$
$$A_{out}^{(+)} = -r^*$$

– We'll use this property shortly.

Let's consider further the idea that $\mathcal{I}^- \cup H^-$ can be regarded as a Cauchy surface for massless KG equation. Consider first the simpler case of flat space. Then for

$$u(x) = f_\ell(r,t)\,\frac{1}{r}\,Y_{\ell m}(\theta,\phi),$$

we have

$$\partial^\mu\partial_\mu u = 0 \Rightarrow \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + V_\ell(r)\right] f_\ell(r,t) = 0$$

where $V_\ell(r) = \dfrac{\ell(\ell+1)}{r^2}$

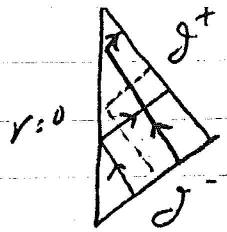
(This is the $M\to 0$ limit of Schwarzschild.)

For e.g. the $\ell=0$ wave, the solution is

$$f_0(r,t) = g(u) + h(v) \qquad \begin{matrix} u = t-r \\ v = t+r \end{matrix}$$

The B.C. u = finite at $r=0$ becomes $f=0$ at $r=0$ or $g(t)+h(t)=0 \Rightarrow$

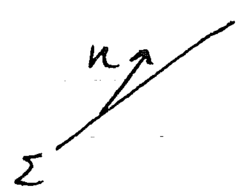
$$f_0(r,t) = g(u) - g(v)$$



Thus, for $l=0$ wave (and in fact for all partial waves) specifying f on $\mathcal{J}^-$ ($u=-\infty$) completely determines f. In particular, there is no need to specify a "normal derivative"; the "normal" actually lies on the null slice. To distort this null surface to a finite u null Cauchy surface, we must include a constant v slice as well for surface to be Cauchy (dotted line in figure). Then specifying "incoming" wave $f(v)$ on u=constant and "outgoing" wave v=constant determines solution completely.

Consider how the KG inner product behaves as the surface Σ on which it is evaluated tips toward light cone

$$(f,g) = i\int_\Sigma d^3x\,\sqrt{h}\;n^\mu\left(f^*\partial_\mu g - \partial_\mu f^* g\right)$$



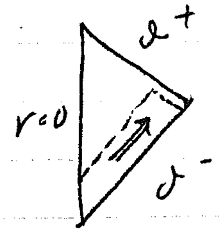
As Σ tips toward null, n^μ tips toward Σ. When Σ becomes null, n^μ blows up (since normalized by $n^\mu n_\mu = 1$), but the divergence is canceled by zero of $\sqrt{h}$

For $\begin{matrix} f \\ g \end{matrix} = \sum_{l,m} \left\{\begin{matrix} f_{l,m}(r,t) \\ g_{l,m}(r,t)\end{matrix}\right\} \frac{1}{r} Y_{lm}(\theta,\phi)$,

we have

$$(f,g) = \sum_{l,m} i\int dv\left(f^*_{l,m}\partial_v g_{l,m} - \partial_v f^*_{l,m} g_{l,m}\right)$$

– evaluated at $u=-\infty$ null surface

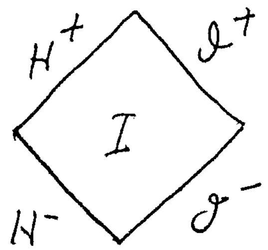


we can, in fact, distort $\mathscr{I}^-$ through the sequence of "Cauchy" null surfaces (e.g. dotted line in the figure) that eventually ends at $\mathscr{I}^+$

we can use as a basis for KG solutions

$$\{\text{solutions}\} \simeq \{\text{incoming on } \mathscr{I}^-\} \simeq \{\text{outgoing on } \mathscr{I}^+\}$$

And we can construct a Fock space in terms of solutions that are positive frequency $f_\ell \sim e^{-i\omega v}$ on $\mathscr{I}^-$ or $f_\ell \sim e^{-i\omega u}$ on $\mathscr{I}^+$.



Similarly, in region I of Kruskal extension, we have

$$\{\text{solutions to (massless) KG eqn}\}$$

$$\simeq \{\text{incoming on } \mathscr{I}^-\} \oplus \{\text{incoming on } H^-\}$$

$$\simeq \{\text{outgoing on } \mathscr{I}^+\} \oplus \{\text{outgoing on } H^+\}$$

that is, form basis for
- solutions that vanish on H^-
- solutions that vanish on $\mathscr{I}^-$

we can construct a Direct product of two Fock spaces

$$\mathcal{H} \simeq \mathcal{H}_{\mathscr{I}^-} \otimes \mathcal{H}_{H^-}$$

or
$$\mathcal{H} \simeq \mathcal{H}_{\mathcal{I}^+} \otimes \mathcal{H}_{H^+}.$$

There is a natural way to choose a basis for solutions with positive KG norm on $\mathcal{I}^-$ or $\mathcal{I}^+$ – the solutions that have positive frequency wrt Schwarzschild time
$$\sim e^{-i\omega v} \text{ on } \mathcal{I}^-$$
$$\sim e^{-i\omega u} \text{ on } \mathcal{I}^+$$

It is less obvious how to choose a basis on H^- or H^+. We could again define positive frequency wrt to the Killing vector $\frac{\partial}{\partial t}$. But on the horizon

$r^* = -\infty$ $\quad \frac{\partial}{\partial U}, \frac{\partial}{\partial V}$ are also Killing

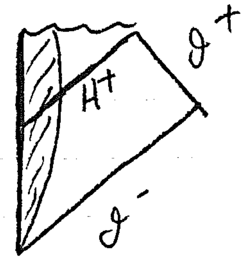
vectors. Defining frequency wrt U, V at the horizon is more natural than defining it wrt u, v,
from the viewpoint of a freely falling local observer, as U, V are the affine parameters of the radial null geodesics, as we have seen.

From the perspective of an observer in region I, this ambiguity in the construction of $\mathcal{H}_{H^+}$ is of little consequence. The quanta in the Fock space $\mathcal{H}_{H^+}$ are quanta that cross the future horizon into region

F (and meet the singularity), thus escaping detection in region I. For the purpose of describing measurements of observables localized in Region I, we may construct a density matrix by summing over $\mathcal{H}_{H+}$ degrees of freedom, and this density matrix does not depend on the $\mathcal{H}_{H+}$ basis. So there is no need to resolve the ambiguity in construction of $\mathcal{H}_{H+}$.

On the other hand, how we define the vacuum state of the $\mathcal{H}_{H-}$ Fock space <u>does</u> affect what a region I observer sees. To describe what she sees, we must resolve the ambiguity and pick out a particular $\mathcal{H}_{H-}$ state. The way to pick this state is to consider, not the Kruskal geometry, but the background of an object that collapses to form a black hole, a background that has <u>no</u> past horizon.

Collapsing Star



In the indefinite past, there is no horizon (and no singularity) So all radial null geodesics, continued back in time, arrive at $\mathscr{I}^-$. Thus, we may regard $\mathscr{I}^-$ as a Cauchy surface for massless KG eqn. At $\mathscr{I}^-$, the potential can be neglected, and a basis for solutions is

$$u_{\omega l m} = \frac{1}{\sqrt{4\pi\omega}}\, e^{-i\omega v}\, \frac{1}{r}\, Y_{lm}(\theta,\phi)$$

and $u^*_{\omega l m}$ — so $(u_{\omega l m}, u_{\omega' l' m'}) = \delta_{ll'}\delta_{mm'}\delta(\omega-\omega')$
$(u, u^*) = 0$

This basis is chosen to be positive frequency with respect to the Schwarzschild time — i.e. the Killing vector $\frac{\partial}{\partial t}$

The corresponding vacuum state $|0\,\mathrm{in}\rangle$, such that

$$a_{\omega l m}|0\,\mathrm{in}\rangle = 0$$

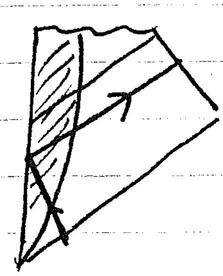
is the state such that no quanta are incoming from $\mathscr{I}^-$ toward the collapsing object.

The question we wish to address is — what does an observer see coming out indefinitely far from the horizon that forms when the star collapses — that is, what quanta arrive at $\mathcal{I}^+$? Our task is to expand the Fock state $|0\,\mathrm{in}\rangle$ in terms of the Fock space.

$$H_{H^+} \otimes H_{\mathcal{I}^+}$$

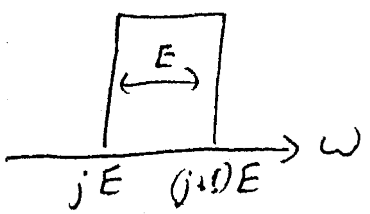
By tracing over the degrees of freedom that propagate to the future horizon, we can obtain a density matrix appropriate for observations of the properties of the outgoing radiation that arrives at $\mathcal{I}^+$

The geometry of the collapsing star is certainly not static, so we have reason to expect particle production to occur. Of course, outside the surface of the star, by Birkhoff's theorem, the geometry of a spherically symmetric collapsing star is static. A wave encounters time-dependent geometry only as it propagates through the collapsing star.



This propagation through the star scatters an incoming wave in a manner that does not preserve its frequency, and in particular mixes up waves that are positive and negative frequency wrt $\partial/\partial t$.

In considering propagation through the star, it is actually much more convenient to use an alternative basis—(radial) wave packet states that are localized in v. That way we can consider waves that arrive at the center $r=0$ of the collapsing star at different times. We construct the packet by superposing radial plane waves with frequencies in a narrow band of width E, and ranging from jE to $(j+1)E$, with j an integer (nonnegative)

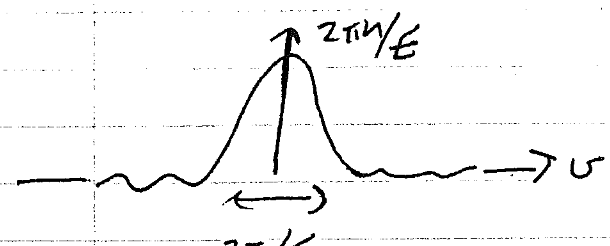


For each partial wave, then, in place of

$$u_\omega \sim \frac{1}{(4\pi\omega)^{1/2}} e^{-i\omega v}$$

we have

$$u_{jn}^{(E)} = \frac{1}{\sqrt{E}} \int_{jE}^{(j+1)E} e^{2\pi i n\omega/E}\, u_\omega\, d\omega$$



This packet has width $\Delta v = 2\pi/E$ and is centered where phase is stationary

$$v = \frac{2\pi}{E} n$$

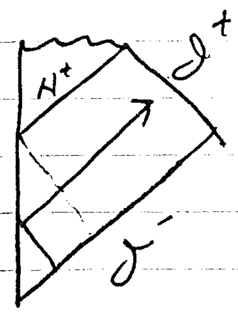
— This basis is obviously complete and orthonormal. We choose E to be small so wave is nearly monochromatic, and is broad (compared to size of body).

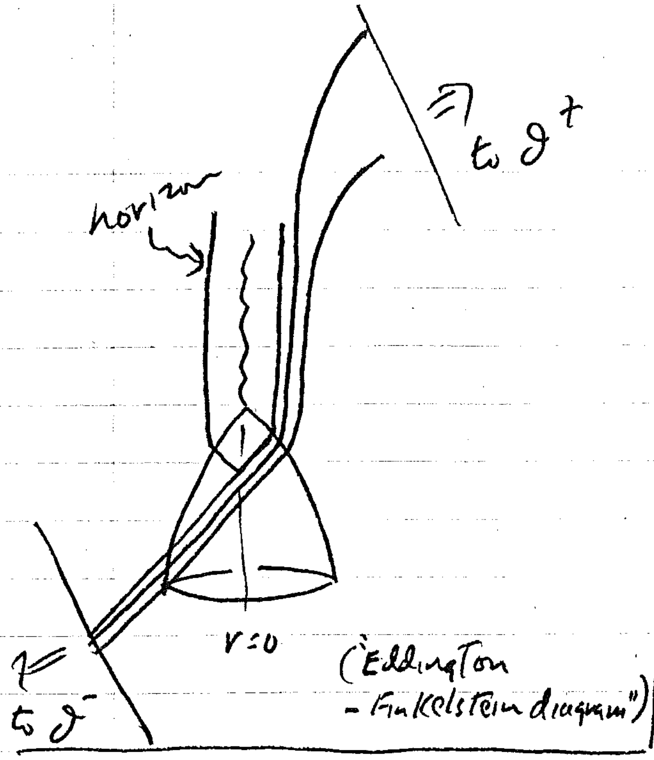
Now we need to propagate these positive frequency wave packets through the collapsing body and out to $\mathcal{I}^+$, where they can be Fourier analyzed with respect to the basis

$$\frac{1}{(4\pi\omega)^{1/2}}\begin{cases} e^{-i\omega u} \\ e^{+i\omega u}\end{cases}$$

for positive and negative frequency (wrt $\frac{\partial}{\partial t}$) solutions of KG at $\mathcal{I}^+$.
This sounds hard, and it seems that the results should depend on the details of the collapse — that is, on the time-dependent radial profile of the spherically-symmetric collapsing star.

This would indeed be so, were it not for the following crucial insight (Hawking). It is indeed true that wave packets that propagate through the collapsing star long before the formation of the horizon are perturbed in a way that depends on the details of the collapse. But these wave packets have nothing to do with the radiance of the black hole long after it forms.





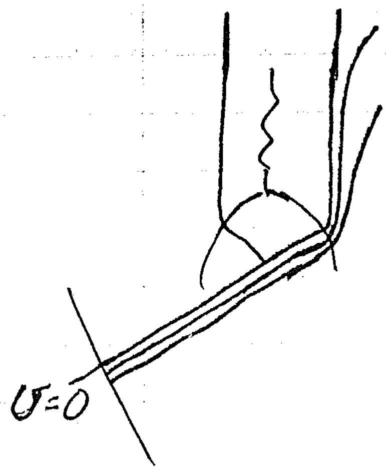
For this purpose, we are instead interested in field modes that reached the center of the star just prior to the formation of the horizon, because these modes are delayed arbitrarily long (in Schwarzschild time) near the horizon. They are wave packets that propagate away from the vicinity of the horizon at arbitrarily late times.

Furthermore, these arbitrarily-long delayed modes experience an arbitrarily large red shift. So a mode of given frequency ω at $\mathcal{I}^+$ has an enormously blue shifted frequency at $\mathcal{I}^-$, if it emerges from the vicinity of the horizon very late. For this reason, the propagation from $\mathcal{I}^-$ through the star and out to vicinity of the horizon can be treated (with arbitrary accuracy) in a "geometrical optics" approximation. In this approximation, the details of collapse do not matter.

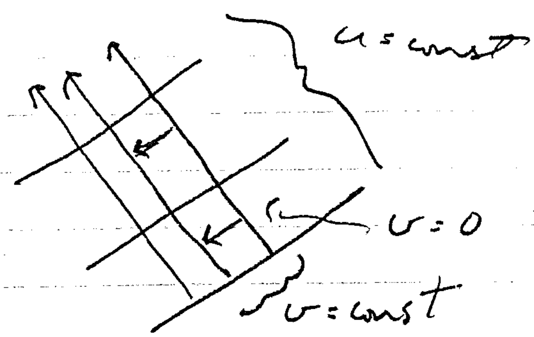
"Geometrical optics" means that the wavelength is very short compared to the characteristic length scale of the potential (WKB approximation) then light waves can be treated as rays. The solution to KG is

$$u \sim e^{i\Theta}$$

where the constant-Θ surfaces are null surfaces (propagate, like particles, at $v = c$).



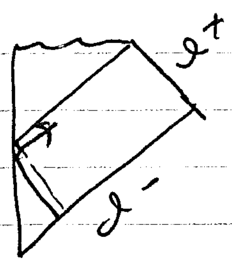
We are interested in waves very close to the radial geodesic (by convention, $v = 0$ at $\mathcal{I}^-$) that gets caught at the horizon. We want to know relation between incoming wave $\sim e^{-i\omega v}$ and outgoing wave that has just passed through the collapsing star and is still near the horizon.



Consider the null geodesic $v = 0$ and another nearby one at $v < 0$. These geodesics are both surfaces of constant phase of an incoming wave $f(v)$.
Curvature may cause these null geodesics to deviate from one another.
But if the geodesics are sufficiently close, deviation is linear. Geodesic deviation can be described as follows: Consider

null surfaces that cross the incoming rays. A vector in this surface points from the $v=0$ geodesic to the neighboring one. That geodesic deviation is linear means that the affine parameter length of this deviation (null vector) changes linearly as a function of affine parameter length of the incoming ($v=$ const) geodesics.

But in geometrical optics, wave f is just a function of the incoming null geodesics, and is hence a linear function of the affine parameter separation between the geodesics. So a wave $f(v)$ on $\mathscr{I}^-$ becomes $f(a\lambda+b)$ as it propagates in.

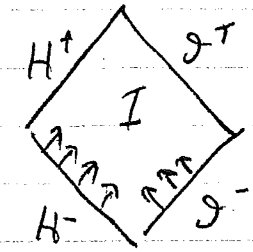


But after the horizon has formed, as the wave is pulling away from the horizon and is still close to the horizon, the affine parameter along the deviation vector is Kruskal null coordinate U, so we have

$$\text{wave} \sim f(aU+b)$$

or for $e^{-i\omega v}$ incoming from $\mathscr{I}^-$ we have $\sim e^{-i\omega aU}$ coming out from horizon.

Therefore, for the purpose of describing the emission at late time from the black hole,



An incoming wave that is positive frequency at $\mathcal{J}^-$ wrt $\partial/\partial t$, and propagates through the collapsing star is equivalent to a wave incoming from past horizon on Region I of Kruskal manifold that is positive frequency wrt $\dfrac{\partial}{\partial U}$.

(This is Unruh's way of rephrasing Hawking's argument.)

This is the most elusive conceptual point in the theory of black hole radiance. With it in hand, it is now reasonably straightforward to proceed with the calculation of the black hole density matrix.

The idea is that we define a vacuum state $|0\,in\rangle$ by

$$a^{H^-}|0\,in\rangle = a^{\mathcal{J}^-}|0\,in\rangle = 0$$

then we calculate Bogolubov coefficients and express $|0\,in\rangle$ in terms of Fock basis of