

$$H_{H^+} \otimes H_{g^+}$$

Finally we trace out the H_{H^+} degrees of freedom to obtain a density matrix that describes outgoing radiation detectable at g^+

It does not really matter how we choose basis for the Fock space H_{H^+} , but we can make the calculation easier and the interpretation of the result clearer by making a sensible choice of basis.

In the description below, I will drop the l, m indices on the solutions and creation and annihilation operators. It is understood that we carry out a similar analysis for each partial wave.

We'll use the obvious bases, pos frequency wrt ∂_t on $\mathcal{I}^\pm$

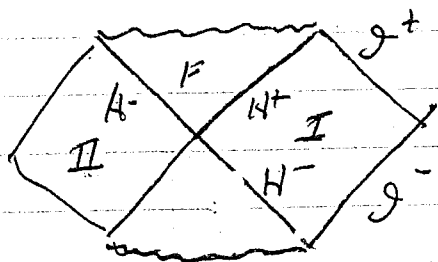
$$u_\omega^{\text{in}} \sim e^{-i\omega v} \text{ on } \mathcal{I}^-$$

$$u_\omega^{\text{out}} \sim e^{-i\omega u} \text{ on } \mathcal{I}^+$$

and for H^+ we'll also use positive frequency wrt ∂_t

$$u_\omega^{H^+} \sim e^{-i\omega v} \text{ on } H^+$$

On H^- we want to choose a basis of solutions that are + freq wrt $\frac{\partial}{\partial U}$
 E.g. $e^{-i\omega' U}$



What we will actually do is make the calculation as similar as possible to our Rindler spacetime calculation by specifying a basis on the extended H^-

We'll have

$$u_{\omega}^{I, H^-} \sim e^{-i\omega u} \Theta(-U) \text{ on } H^-$$

$$u_{\omega}^{II, H^-} \sim e^{i\omega u} \Theta(U) \text{ on } H^-$$

— these are positive freq wrt $\frac{\partial}{\partial t}$;
 note the sign change, since time runs backwards in region II

Now we construct solutions that are positive frequency wrt Kruskal time by demanding analyticity in U in the lower half plane. Thus

$$u_{1, \omega}^{\text{Kruskal}} = N_{\omega} (u_{\omega}^{I, H^-} + e^{-4\pi M \omega} u_{\omega}^{II, H^-*})$$

$$u_{2, \omega}^{\text{Kruskal}} = N_{\omega} (u_{\omega}^{II, H^-} + e^{-4\pi M \omega} u_{\omega}^{I, H^-*})$$

where $N_{\omega} = (1 - e^{-8\pi M \omega})^{-\frac{1}{2}}$

Here we have used

$$u = -4M \ln(-U) \quad \text{Region I}$$

$$u = 4M \ln(U) \quad \text{Region II}$$

and have continued around the cut of the log by giving U a negative imaginary part, as on page (3.27). We'll use these as our basis for the solutions on H^-

So our "in" solutions are

$$u_{1,w}^{\text{Kruskal}}, u_{2,w}^{\text{Kruskal}}, u_w^{\mathcal{I}^-}$$

And our in vacuum will satisfy

$$0 = a_{1,w}^{\text{Kruskal}} |0, \text{in}\rangle = a_{2,w}^{\text{Kruskal}} |0, \text{in}\rangle = a_w^{\mathcal{I}^-} |0, \text{in}\rangle$$

We thus impose a condition on the fields at H^- in II, but there is no harm in doing so, since this will have no effect on what is seen in region I.

For our "out" solutions, we'll use

$$u_w^{\mathcal{I}^+}, \underbrace{u_w^{H^+}, u_w^{\text{II}, H^-}}_{\text{arbitrary choices}}$$

arbitrary choices

Now, to derive Bogolubov transformation, we must first take into

account the scattering off the potential barrier. we may define transmission and reflection coefficients by

$$u_{\omega}^{I, H^{-}} = t_{\omega} u_{\omega}^{I^{+}} + r_{\omega} u_{\omega}^{H^{+}}$$

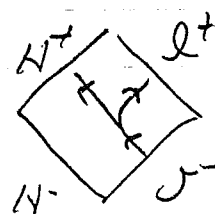
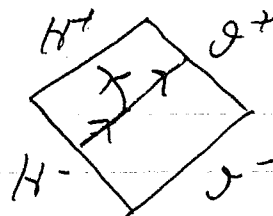
then unitarity implies

$$|t_{\omega}|^2 + |r_{\omega}|^2 = 1,$$

and by time reversal

$$u_{\omega}^{I^{-}} = -r_{\omega}^{*} u_{\omega}^{I^{+}} + t_{\omega}^{*} u_{\omega}^{H^{+}}$$

(up to an unimportant overall phase) - see page (4.27)



Now we can express

$$u_{\omega}^{in} = \begin{pmatrix} u_{1,\omega}^{Kruskol} \\ u_{2,\omega}^{Kruskol} \\ u_{\omega}^{I^{-}} \end{pmatrix} \text{ in terms of } u_{\omega}^{out} = \begin{pmatrix} u_{\omega}^{I^{+}} \\ u_{\omega}^{H^{+}} \\ u_{\omega}^{I, H^{-}} \end{pmatrix}$$

we have

$$u_{1,\omega}^{Kruskol} = N_{\omega} \left[t_{\omega} u_{\omega}^{I^{+}} + r_{\omega} u_{\omega}^{H^{+}} + e^{-4\pi M \omega} u_{\omega}^{I, H^{-}} \right]$$

$$u_{2,\omega}^{Kruskol} = N_{\omega} \left[u_{\omega}^{I, H^{-}} + e^{-4\pi M \omega} (t_{\omega}^{*} u_{\omega}^{I^{+}*} + r_{\omega}^{*} u_{\omega}^{H^{+}*}) \right]$$

$$u_{\omega}^{I^{-}} = (-r_{\omega}^{*} u_{\omega}^{I^{+}*} + t_{\omega}^{*} u_{\omega}^{H^{+}*})$$

or

$$u_w^{\text{in}} = \alpha_w u_w^{\text{out}} + \beta_w u_w^{\text{out}*}$$

$$\alpha_w = \begin{bmatrix} N_w t_w & N_w r_w & 0 \\ 0 & 0 & N_w \\ -r_w^* & t_w^* & 0 \end{bmatrix}$$

$$\beta_w = \begin{bmatrix} 0 & 0 & N_w e^{-4\pi M_w} \\ N_w e^{-4\pi M_w} t_w^* & N_w e^{-4\pi M_w} r_w^* & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we have $\alpha_w^{-1} = \begin{bmatrix} N_w^{-1} t_w^* & 0 & r_w \\ N_w^{-1} r_w^* & 0 & t_w \\ 0 & N_w^{-1} & 0 \end{bmatrix}$

$$(\alpha^{-1}\beta)^* = \begin{bmatrix} 0 & 0 & e^{-4\pi M_w} t_w \\ 0 & 0 & e^{-4\pi M_w} r_w \\ e^{-4\pi M_w} t_w & e^{-4\pi M_w} r_w & 0 \end{bmatrix}$$

- symmetric as it must be.

Now compare page (2.24): $\begin{pmatrix} u' \\ u'^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} u \\ u^* \end{pmatrix}$

and page (2.31):

$$|0'\rangle = \text{phase} |\det \alpha|^{-\frac{1}{2}} \exp \left[\frac{1}{2} a^\dagger (\alpha^{-1}\beta)^* a^\dagger \right] |0\rangle$$

So -- we have

$$|0, in\rangle = \prod_{\omega} N_{\omega}^{-1} \exp \left[e^{-4\pi M \omega} (a_{\omega}^{\Pi, H^-})^{\dagger} [t_{\omega} (a_{\omega}^g)^{\dagger} + r_{\omega} (a_{\omega}^{H^+})^{\dagger}] \right] |0, out\rangle$$

($\det \alpha_{\omega} = -N_{\omega}^2$) - (Recall here is a product over ℓ, m also, which has been suppressed)

If we trace over the $H^{\pm}, \Pi$ Fock space, we find

$$\rho_I = \sum_{\Pi} \langle \Psi_{\Pi} | 0, in \rangle \langle 0, in | \Psi_{\Pi} \rangle$$

(and is, $|0\rangle = \sum_{\Psi} |\Psi\rangle \langle \Psi| \Rightarrow \rho_I = \sum_{\Psi} |\Psi\rangle \langle \Psi|$)

or

$$\rho_I = \prod_{\omega} (1 - e^{-8\pi M \omega}) \sum_{n_{\omega}} e^{-8\pi M n_{\omega}}$$

$$\frac{1}{n_{\omega}!} [t_{\omega} (a_{\omega}^g)^{\dagger} + r_{\omega} (a_{\omega}^{H^+})^{\dagger}]^{n_{\omega}} |0, out, I\rangle$$

$$\langle 0, out, I | [t_{\omega}^* (a_{\omega}^g) + r_{\omega}^* (a_{\omega}^{H^+})]^{n_{\omega}}$$

We can proceed to trace over the states of H_{H^+} to find ρ_I , i.e.

— which describes the BH radiation seen at ∞ .

But the interpretation of the result is particularly clear in its current form. If $t=1$ and $r=0$, the modes have precisely a Fermi distribution.

When transmission and reflection are taken into account, we see that each mode emerges from the post horizon thermally populated - has amplitude t_ω of being transmitted through the barrier to $\mathcal{I}^+$, and amplitude r_ω for being reflected back to $\mathcal{H}^+$.

We may now take trace wrt $\mathcal{H}^+$ degrees of freedom (an exercise) and find a density matrix that describes the radiation received at $\mathcal{I}^+$; it is

$$\rho_{\mathcal{I}^+} = \prod_{\omega} \left[\sum_{n_{\omega}} |n_{\omega}, \mathcal{I}^+\rangle P_{n,\omega}^{(\beta)} \langle n_{\omega}, \mathcal{I}^+| \right]$$

where

$$P_{n,\omega}^{(\beta)} = (1 - e^{-\beta\omega}) \frac{(e^{-\beta\omega} |t_\omega|^2)^n}{[1 - |r_\omega|^2 e^{-\beta\omega}]^{n+1}}, \quad \beta = 8\pi M.$$

Just thermal distribution for $|t_\omega|^2 = 1$, with $T = \frac{1}{8\pi M}$.

We may also compute the mean occupation number of each mode at $\mathcal{I}^+$ using formula from chapter 2

Comparing with page 2.25,

$$u^{\text{in}} = \alpha u^{\text{out}} + \beta u^{\text{out}*}$$

$$\Rightarrow \begin{aligned} a^{\text{out}} &= \alpha^T a^{\text{in}} + \beta^T a^{\text{in}*} \\ a^{\text{out}*} &= \beta^T a^{\text{in}} + \alpha^T a^{\text{in}*} \end{aligned}$$

And so

$$N_{\omega,i} = \langle 0_{in} | a_{\omega,i}^{out\dagger} a_{\omega,i}^{out} | 0_{in} \rangle$$

$$= \beta_{\omega,i,j}^T \beta_{\omega,i,k}^T \langle 0_{in} | a_{\omega,k}^{in\dagger} a_{\omega,j}^{in\dagger} | 0_{in} \rangle$$

$$= \sum_j \beta_{\omega,j,i}^* \beta_{\omega,j,i} = \sum_j |\beta_{\omega,j,i}|^2$$

From 1st column of β_{ω} matrix on page 4.45, we find

{ i,j are reversed here compared to page 7.33, because in and out have been interchanged. }

$$N_{\omega,j} = |\epsilon_{\omega}|^2 \frac{1}{\beta_{\omega} - 1}$$

- thermal occupation number modified by transmission probability.

The temperature of the emitted radiation, with appropriate constants restored, is

$$kT = \frac{\hbar c^3}{8\pi G M}$$

Numerically

$$T = 6 \times 10^{-8} \left(\frac{M_{\odot}}{M} \right) \text{ } ^{\circ} \text{K}$$

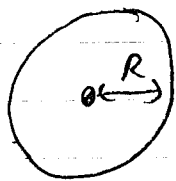
where $M_{\odot} \approx 2 \times 10^{33} \text{ g}$ is solar mass. A solar mass black hole has Schwarzschild radius

$$R = \frac{2GM}{c^2} \sim 3 \text{ km}$$

- The typical wavelength of emitted radiation received at

$\mathcal{I}^+$ is somewhat larger than this.

Since no radiation is coming in from $\mathcal{I}^-$, and a steady flux is received at $\mathcal{I}^+$, the black hole is evidently radiating (even long after the formation of the horizon). To compute the flux, we need only integrate over occupation numbers against a density of states factor.



To count states, imagine putting the BH in a large spherical cavity with radius $R \gg M$. For each l, m , consider outgoing spherical waves
 $\sim e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{r}}$

If walls of the cavity are perfectly reflecting, radial modes are $\sin(\frac{\pi k}{R} r)$ - (we ignore effective potential if cavity is big) and $\Sigma \sim \frac{R}{\pi} \int dK$. But only half of these modes are outgoing. So

$$\Sigma_{\text{outgoing}} \sim \frac{R}{2\pi} \int dK$$

If the modes travel at velocity v , we obtain

$$\text{outgoing flux} = \sum_{l,m} \int \frac{vK}{R} \frac{R}{2\pi} dK N_{l,m}$$

For our massless modes, $v=c$ and $K=\omega$,

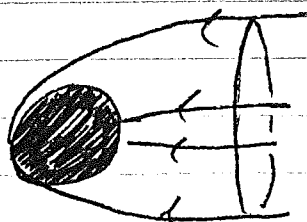
$$\frac{\text{Number Outgoing}}{\text{time}} = \sum_l (2l+1) \int \frac{d\omega}{2\pi} \frac{|\epsilon \omega e|}{e^{\beta\omega} - 1}$$

To find the total luminosity of emitted energy, we weight the modes by the energy ω of the quanta, so

$$L = \frac{\text{energy emitted}}{\text{time}} = \sum_l (2l+1) \int \frac{d\omega}{2\pi} \omega \frac{1}{e^{\beta\omega} - 1}$$

To compute the luminosity, we must find the ℓ 's; this requires numerical solution of the radial KG eqn.

Note that the black hole is a rather unconventional black body, in that the typical wavelength of the emitted radiation is comparable to its size. So it is not really so black — a typical Komol quantum incident on the (centrifugal and curvature) barrier around the black hole may fail to be absorbed. Correspondingly, the emission of the black hole is suppressed, relative to an ideal black body, and long wavelengths.

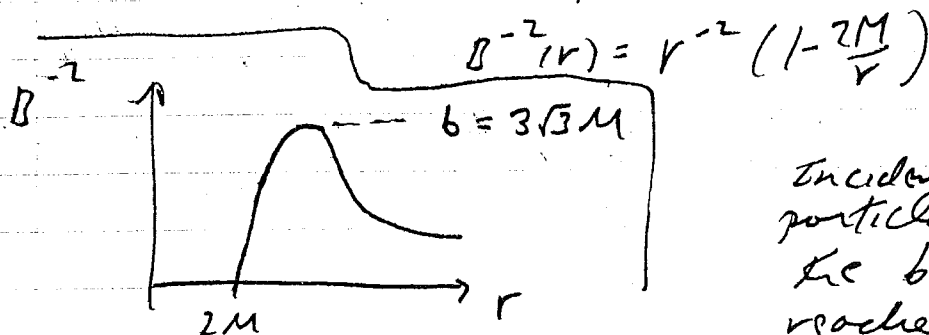


But if the ω of the incoming quantum is high compared to $\frac{1}{r_h}$, then propagation is well described by geometrical optics — the absorption cross section can be computed by considering radial geodesics in Schwarzschild geometry.

See MTW, page 674:

Radial motion of massless particle governed by

$$\left(\frac{dr}{dt}\right)^2 + \underbrace{B^{-2}(r)}_{\text{effective potential}} = \underbrace{b^{-2}}_{\text{impact parameter}}$$



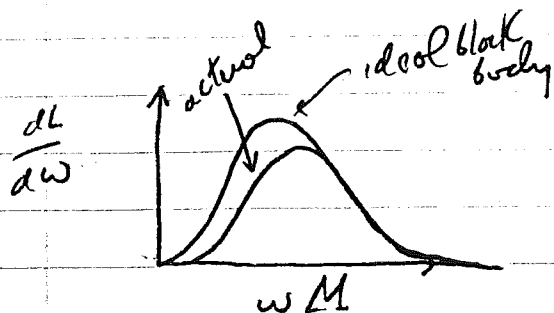
Incident massless particle passes over the barrier and reaches horizon if impact parameter is $b > 3\sqrt{3}M$

So (geometrical optics) cross section for capture of a massless particle is

$$\sigma = \pi (3\sqrt{3}M)^2 = 27\pi M^2$$

For large ω , then, differential luminosity of BH should agree with that of a black body with this area and $\beta = 8\pi M$, namely
(actually $4\pi b_{\text{crit}}^2 = 45$)

$$\frac{dL}{d\omega} = (\text{Area}) \frac{1}{2\pi^2} \frac{\omega^3}{e^{\beta\omega} - 1} \quad (\omega M \gg 1)$$



- but suppressed relative to ideal black body at small ω

Anyway, the ideal black body curve can be used to understand how L scales with M , namely

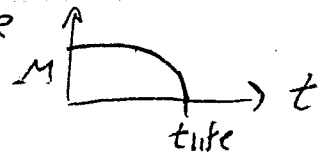
$$L \sim (\text{Area}) T^4 \sim M^2 M^{-4} \sim M^{-2}$$

— the black hole heats up sufficiently as it loses mass that increased temperature more than compensates for loss of area

Integrate $\frac{dM}{dt} = -CM^{-2} \Rightarrow \frac{1}{3}(M^3 - M_0^3) = -C(t - t_0)$
 or $M = [M_0^3 - 3C(t - t_0)]^{\frac{1}{3}}$

— the BH disappears completely in time

$$t_{\text{life}} = M^3 / 3C$$



To calculate C requires numerical calculations (e.g. D. Page — see Chapter 8 of 'The membrane paradigm').
 For $T_{\text{BH}} \ll M_{\text{electron}}$ (or $M_{\text{BH}} \gg 10^{17}$ gram) only light species are emitted

gravitons

photons

neutrinos (if sufficiently light, which we don't know)

Assuming 3 species of effectively massless v's, Page's calculation gives

$$t_{\text{life}} \sim 10^{10} \text{ yr} \left(\frac{M}{5 \times 10^{14} \text{ g}} \right)^3$$

"Primordial" black holes in this mass range would be disappearing today.

Radiation emitted by evaporating primordial black holes has not been seen.

(Note: ν 's actually dominate t_{life} is larger by about a factor of 7-8 if there are no massless ν 's)

$$\text{Or } t_{\text{life}} \sim 10^{66} \text{ yr} \left(\frac{M}{M_0} \right)^3$$

- so solar mass BH's will not disappear for a while. (They are accreting 3°K CBR photons now, but will eventually be colder than CBR radiation, if universe is open.)

Interpretation

Let's consider further why a black hole emits a steady flux of quanta.

It should be evident that the horizon is an absolutely essential ingredient. E.g. suppose we consider the collapse to form a (zero-temperature) neutron star — i.e. a cold static ball somewhat larger than its Schwarzschild radius. During collapse, the geometry inside the ball is not static, and we will not be surprised to see (solar) quanta emitted (No gravitons, though, in spherically symmetric collapse.) But once the star settles down, there will be no emission at late times. In the BH case, there are outgoing

field modes that are delayed arbitrarily long by the horizon, and these give rise to the emission at very late times.

If we consider the spacetime of a static neutron star, it is globally static, both outside and inside the star. So ∂_t is a globally timelike Killing vector — unlike the BH case, where ∂_t tips over and becomes spacelike at the horizon. So there is a natural vacuum

$$|0_{in}\rangle = |0_{out}\rangle$$

- The Fock space states with no quanta at $\mathcal{I}^-$ and no quanta at $\mathcal{I}^+$ are the same state. No particle production occurs in the BG of a neutron star (there is no post horizon H^- to act as the "source" of outgoing particles that reach $\mathcal{I}^+$)

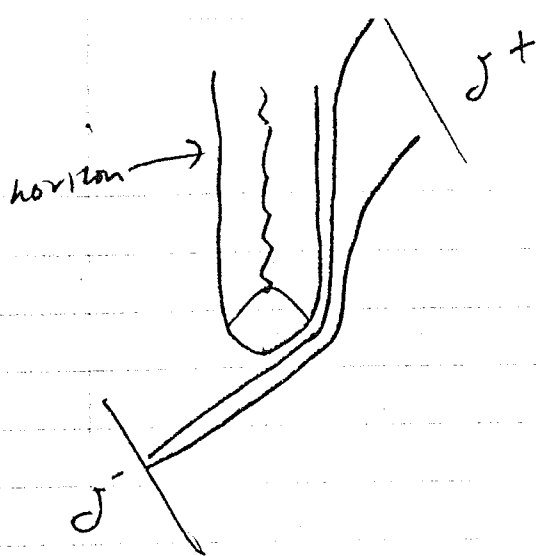
We can imagine squeezing the neutron star until its radius reaches its Schwarzschild radius, and it becomes a black hole. In this way, we can, by quasistatically perturbing the quantum field vacuum of the neutron star background, produce a quantum state in the BH background that is pure vacuum outside the horizon — no quanta emerging from post horizon.

Formally, this is the state obtained by using a basis for the KG equation that is pos. freq w.r.t Schwarzschild time t both on J^- and on H^- (It is called the "Boulware Vacuum".)

There is clearly something unphysical about this state, for it is the vacuum of the fields in the BG of a star that has ∞ proper acceleration at its surface. What would really happen if we squeezed a neutron star so that, when its surface was close enough to the Schwarzschild radius, it would be unable to support itself and would (freely) fall in. The quasistatic approximation would break down, and we would be back to considering a BH that forms from a collapsing object.

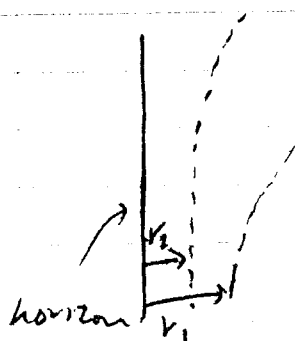
We'll appreciate better the "unphysical" nature of the Boulware vacuum a little later, when we compute the (renormalized) energy-momentum tensor in this state — it diverges at the horizon, even as measured by freely-falling observer.

Notice something funny about our picture of BH radiance. The outgoing flux



long after collapse is alleged to be due to field modes that spend a long time near the horizon, and become strongly redshifted before escaping. But then outgoing

quanta with $\omega \sim 1/r$ at J^+ must be occupying modes that have truly incredible frequencies on J^-



to estimate this consider two photons at radial coordinates r_1, r_2 around time horizon lines, with $r_{1,2} \sim 2M$.

If on outgoing radial null geodesics, these photons propagate as

$$r_* = t + \text{const}$$

and arrive at $r \gg 2M$ redshifted by the factor

$$(g_{00})^{1/2} = \left(1 - \frac{2M}{r_{1,2}}\right)^{1/2} \sim e^{r_*/4M}$$

If initially, frequencies are $\omega_{1,0}, \omega_{2,0}$, then redshifted frequencies are

$$\frac{\omega_1(\infty)}{\omega_2(\infty)} = \frac{\omega_{1,0} e^{r_{*,1}/4M}}{\omega_{2,0} e^{r_{*,2}/4M}} = \frac{\omega_{1,0}}{\omega_{2,0}} e^{8t/4M}$$

where $8t$ is the time interval between receipt of photons 1 and 2 by FIDO.

So if $\omega_1(\infty) \sim \omega_2(\infty)$, then
 $\omega_{2,0} \sim \omega_{1,0} e^{st/4M}$

- the photons that arrive late had exponentially large frequencies coming in.

Of course, there are no physical photons with enormous frequencies. These very high frequency modes are unoccupied, as the initial state was the vacuum at J^- . The point of our calculation of the Bogolubov coefficients is that these modes are not purely positive frequency w.r.t u or v at J^+ if they are positive frequency w.r.t u or v at J^- .

Still, there is something unpleasant about considering frequencies that are enormous, even compared to the Planck scale, at J^- . It should not be necessary to do this. We ought to be able to impose a cutoff on the frequency of the vacuum field fluctuations

$$\omega < \Lambda,$$

and this ought not to drastically modify the calculation of BH radiance as long as $\Lambda \gg T$ (- i.e. temperature of radiation at J^+)

The whole idea of renormalization in quantum field theory is that, for the purpose of discussing physics at low energies (e.g. emitted quanta with temperature $T \ll M_{\text{Planck}}$) we can "integrate out" very short wavelength field fluctuations, and incorporate the effects of these fluctuations into the parameters of an "effective field theory" with a smaller cutoff — these parameters correspond to the quantities that can be measured directly at low-energy ("renormalized," as opposed to "bare" quantities). The idea of "decoupling" of the short wavelength degrees of freedom is that, to good accuracy, the effects of the short-wavelength fluctuations can be completely absorbed into such renormalizations (and that the number of resulting renormalized parameters is small and manageable).

This is, after all, what makes physics possible — we would be in trouble if we needed to know all about Planck scale physics to predict e^+e^- annihilation at 90 GeV. Conversely, we cannot expect to learn much about Planck scale physics by measuring low-energy processes.

So there ought to be an effective field theory description of BH radiance, in which $\omega > \Lambda$ is integrated out and never considered.

Something like this is achieved by the "membrane paradigm" viewpoint of Thorne et al. In effect, they integrate out physics very close to the horizon, and incorporate this short distance physics into a modification of the boundary conditions satisfied on a "stretched horizon" that hovers above the actual horizon.

We may choose the (past) stretched horizon so that the local temperature (measured by a FIDO) here is $T_{\text{local}} \sim \Lambda$.

This membrane then provides an inexhaustible supply of upward propagating quanta occupying the field modes that are pos. freq wrt u on $\mathcal{I}^+$.

It seems to me, though, that a fully satisfactory "effective field theory" picture of a black hole that forms in gravitational collapse remains to be formulated. Perhaps the key to its formulation is the recognition that these super-high frequencies are in a sense fictitious, as they are frequencies measured by (unphysical) FIDO's very close to the horizon.

There should be an alternative language in which to describe the origin of the BH radiation. In our calculation, we used the Bogolubov transformation viewpoint — modes behaving like $e^{-i\omega U}$ on $\mathcal{I}^-$, after propagating through the collapsing star just ahead of formation of horizon, behave like $e^{-i\omega U}$ on (stretched) post horizon — and then, when expanded in terms of the $e^{-i\omega u}$ basis on $\mathcal{I}^+$, can be either pos or neg. frequency.

The alternative viewpoint is that the collapsing star is a time dependent classical source that is coupled to the field modes that are positive frequency wrt t . If the source varies nonadiabatically on a time scale $\sim \omega^{-1}$, it is capable of exciting quanta with frequency $\sim \omega$.

To produce a steady flux of outgoing quanta at late times, the collapsing star must be producing shorter and shorter wavelength quanta as its surface gets closer and closer to the event horizon — so that after the substantial red shift that occurs as they pull away from the horizon, they emerge with nonnegligible frequency.

The idea is that, during each e-folding

of the red-shift factor $e^{r_x/4M}$ as the surface collapses toward $r_x = -\infty$ (the horizon), the proper distance from the surface to the (incipient) horizon changes by a factor of order one, and quanta with this wavelength $\sim e^{r_x/4M}$ are excited by the star. When they reach $r \gg 2M$, the red-shifted wavelengths are order-one.

So, as the star collapses, the collapse excites quanta of higher and higher frequency as the surface approaches the horizon, in such a way that the red shifted frequency of the quanta detected at ∞ remains fixed.

(This viewpoint is worked out in detail in U. H. Gerlach, PRD 14 (1976) 1479.)

What do FIDO's and FFO's see, close to the BH horizon?

The first important point is that the temperature close to the horizon, as measured by a FIDO is

$$T(r) \sim \frac{1}{8\pi M} \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}$$

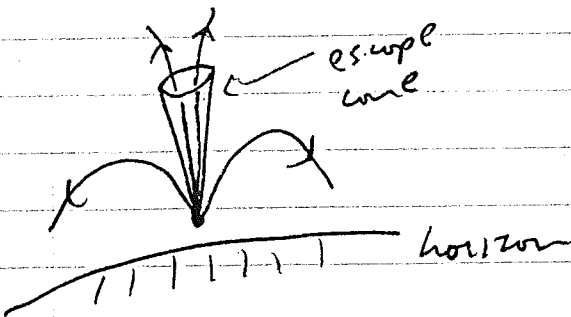
$$= \frac{1}{2\pi \kappa_{\text{proper}}}$$

where S_{proper} is the proper distance to the horizon. (This is just as for a Rindler observer in the Minkowski vacuum.) Hence, this observer cannot regard the radiation as streaming out from the horizon - it is not describable in the language of geometrical optics over a distance comparable to the distance to the horizon.

In fact, to an approximation that becomes better and better for a FIDO closer and closer to the horizon, the FIDO sees a (nearly) isotropic thermal flux. This is because nearly all of the radiation is unable to penetrate the effective potential

$$V_e = \left(1 - \frac{2M}{r}\right) \left(\frac{L^2}{r^2} + \frac{2M}{r^3}\right)$$

and gets reflected back to the (future) horizon.



We gain a heuristic understanding of this by considering the propagation of the outgoing wave in geometrical optics.

An outgoing (null) geodesic manages to escape from the black hole only if directed in a narrow "escape cone" that gets narrower and narrower closer and closer to the horizon.

As measured by FIDO, half opening angle of cone must be

$$\delta \approx \frac{3\sqrt{3}}{2} \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \quad \text{for } r \gg 2M$$



(-calculated in MTW, page 675) for null geodesic to escape.

In terms of wave propagation, there are propagating modes for given ω and all l as $r^* \rightarrow -\infty$, but almost all are reflected back by the centrifugal barrier, just as almost all l -modes "miss" the BH when incoming from $r = \infty$ (time-reversed process)

The reflected quanta rain back down onto the FIDO's near the horizon, endowing the black hole with what Thorne ("Membrane Paradigm") calls a "thermal atmosphere". Thus our analogy with a Rindler observer in the Minkowski vacuum becomes very accurate close to the horizon. And since FIDO's very close to the horizon see a thermal bath, we know that FFO's at the horizon see vacuum.

Massive Particles

How is the analysis modified if our scalar field has a mass?

Now that we see that the result can be interpreted in terms of a flux of thermal quanta emerging from the post horizon with the local (FIDO-measured) temperature $T(r) = \frac{1}{8\pi M} \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}$,

we can see that the mass should have no effect, because it can be neglected close to the horizon. The only difference between massless and massive particles is their ability to penetrate the barrier.

For a massive scalar field, the effective potential is

$$V_e = \left(1 - \frac{2M}{r}\right) \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2 \right]$$

It turns off (effectively, because frequencies are blue-shifted) close to the horizon, and

$$V_e \rightarrow m^2 \quad \text{for } r \rightarrow \infty.$$

Modes with $\omega^2 < m^2$ are unable to propagate and are completely reflected. Those modes that escape (to i^+) have a thermal distribution, modified by a transmission probability $|t|^2$.

(The analysis on page 4.25 is slightly modified, because frequencies at $r_* \rightarrow \pm\infty$ do not match, but probability conservation and time-reversal invariance still hold, and the result remains valid; outgoing and incoming waves are reflected with the same probability.)

Kerr Black Hole

Let's next consider how our calculation of black hole radiance is modified when we consider the (more realistic) case of a rotating Kerr black hole.

The Kerr black hole is the end result of non-spherical gravitational collapse. When spherical symmetry is not assumed, the exterior geometry of a collapsing object may be complicated (e.g., many gravitational multipoles, etc.) However, once a horizon forms, we may invoke the remarkable black hole uniqueness theorems, which tell us that the only stationary black hole solutions to the (vacuum) Einstein equation are the Kerr solutions. (Here we are actually making an additional assumption - the cosmic censorship hypothesis - that the geometry exterior

to the horizon is globally hyperbolic, for this is a necessary hypothesis in the no-hair theorems.)

Now the idea is that, after collapse, the object that forms loses all of its "hair" by radiating it to infinity or through the (future) horizon, and settles down to a stationary configuration. Thus the exterior geometry becomes extremely well approximated by Kerr geometry (to exponential accuracy, as time passes).

The Kerr solutions are a two-parameter family of solutions, parametrized by

$$\begin{aligned} M & - \text{mass} \\ a & - \text{angular momentum/mass} \end{aligned}$$

In the Boyer-Lindquist coordinate system:

$$ds^2 = \frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 - \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a dt]^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta$$

$$\Delta = r^2 - 2Mr + a^2$$

$a = 0$ is the Schwarzschild solution

This metric is stationary (independent of t) but is not static ($g_{0i} \neq 0$). It is invariant not under $t \rightarrow -t$, but has instead the symmetry

$$t \rightarrow -t \\ \phi \rightarrow -\phi$$

as time-reversal changes the BH's sense of rotation.

In the Schwarzschild case, we were able to invoke the Rindler analogy — We considered FIDO close to horizon, and, since she has proper acceleration

$$a_{\text{proper}} = \frac{1}{4M} \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}$$

she should see radiation with temperature

$$T(r) = \frac{a(r)}{2\pi}.$$

To invoke the Rindler analogy in the Kerr case, we need to decide who the FIDOs are. This is not so obvious, because the rotating BH drags locally inertial frames along as it spins.

Continuing to use the coordinates t, r, θ, ϕ , we may rewrite the Kerr metric in the form

$$ds^2 = \alpha^2 dt^2 - g_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt)$$

α "lapse" β "shift"

The world line of FIDO is chosen so that she perceives the geometry to be static - independent of time, and with constant - time surfaces orthogonal to her motion. Thus, we choose

$$\left(\frac{d\vec{x}}{dt}\right)_{\text{FIDO}} = -\vec{\beta}$$

and then $\left(\frac{d\tau}{dt}\right)_{\text{FIDO}} = \alpha$

is the red shift factor that specifies the rate at which a FIDO's clock runs.

(See "Membrane paradigm", p. 67 ff.)

The horizon is the surface where $\alpha = 0$.

This occurs for

$$r = r_H = M + \sqrt{M^2 - a^2} \quad \left\{ \begin{array}{l} a < M, \text{ according} \\ \text{to cosmic} \\ \text{censorship} \end{array} \right.$$

The FIDOs spin around the BH, with an angular velocity Ω , as the horizon is approached becomes

$$\omega_{\text{FIDO}} \rightarrow \Omega_H = \frac{a}{2Mr_H}, \text{ as } r \rightarrow r_H$$

In fact, because of the freezing of the motion (in Boyer-Lindquist "universal" time t) as the horizon is approached, all timelike and null paths corotate at Ω_H at the horizon.

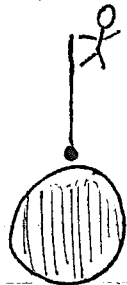
Now, the proper acceleration, relative to FIDO, of FIDO very close to the horizon is

$$a_{\text{FIDO}} \approx \frac{1}{\alpha} K$$

where K is the "surface gravity"; it turns out to be a constant on the horizon, given by

$$K = \frac{r_H - M}{2Mr_H}$$

K is acceleration measured per unit t , rather than per unit of FIDO proper time.



We can interpret it this way: If a mass m is hovering over the horizon, at rest in the FIDO frame, a distant observer supporting the mass at the end of a long string must exert a force $F = mK$.

If we now invoke the Rindler analogy for these FIDOs, we conclude that they detect radiation with the locally measured temperature

$$T(r, \theta) = \frac{1}{2\pi(r, \theta)} K \left(\frac{1}{2\pi} \right)$$

which becomes

$$T_\infty = \frac{K}{2\pi} \quad \text{at } r = \infty$$

- So $\frac{1}{4M}$ is replaced by K when we generalize our Schwarzschild analysis to Kerr.

The Calculation

If we repeat our calculation of the Bogolubov transformation for a Kerr black hole, the most significant difference is that

$$\frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi}$$

is the Killing vector, in the Kerr case, that generates the horizon — i.e., this generates the world line of an outgoing photon that is stuck at the horizon.

In the Schwarzschild case, the key to the derivation of the Bogolubov transformation was the relation between the Killing parameter u and the affine parameter U of a null geodesic at the horizon

$$U = -e^{-u/4M}$$

In the Kerr case, this becomes, for an outgoing geodesic at the horizon

$$(\text{Affine Parameter}) = -\exp[-K(\text{Killing parameter})]$$

— with K replacing $\frac{1}{4M}$

In addition, since an incoming wave

$$u \sim e^{-i\omega t} e^{i\mu\phi}$$

has eigenvalue

$$\omega - \Omega_{\text{HM}}$$

wrt the Killing vector $\frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi}$ that generates the horizon, when the Bogolubov coefficients are calculated, the factor

$$e^{-8\pi M\omega} \text{ becomes generalized to } e^{-\frac{2\pi}{K}(\omega - \Omega_{\text{HM}})}$$

The occupation nos. of the outgoing radiation, therefore, are

$$n_{\omega m} = |t_{\omega m}|^2 \frac{1}{e^{\beta(\omega - \Omega_{\text{HM}})} - 1}$$

with $\beta = 2\pi/K$. Thus, $\beta\Omega_H$ enters the distribution like a chemical potential — as expected for a rotating thermal reservoir.

A peculiar feature of the result is — for $\omega - \Omega_{\text{HM}} < 0$, we have

$$n = |t|^2 (\text{negative occupation number})$$

What really happens is that $|t|^2$ should be replaced by $1 - |r|^2$, and for $\omega - \Omega_{\text{HM}}$ we have

$$1 - |r_{\omega m}|^2 < 0$$

This is the phenomenon called "superradiance." The amplitude of the wave that is reflected by the potential ^{barrier} is actually larger than the amplitude of the incoming wave. In the process, the reflected wave has picked up some angular momentum from the BH. In quantum-mechanical language, this classical field theory process can be described as "stimulated emission." There is, in addition, a spontaneous emission process whereby the BH reduces its angular momentum by emitting a quantum with $m\Omega_H > \omega$ (this is not Hawking radiation, but a logically independent phenomenon associated with the ergosphere of the Kerr black hole.) The rate for this process is enhanced by incoming quanta.

(More detail concerning the mode expansion in the Kerr background can be found in S. DeWitt, Phys. Rep. 19C (1975) 295.)

Since time-reversal invariance is broken by the Kerr background, the relation between reflection of modes incoming from H^- and from J^- is somewhat different than for the Schwarzschild background.

The symmetry of Kerr is $t \rightarrow -t$
 $\phi \rightarrow -\phi$,

which relates $\omega \rightarrow -\omega$ to $m \rightarrow -m$.
 That is, relates reflection of mode with
 angular momentum m coming from H^+
 to reflection of mode with angular
 momentum $-m$ coming from I^- .
 Hence, for $\Omega_H > 0$, we have

$\left\{ \begin{array}{l} m > 0 \text{ preferentially emitted} \\ m < 0 \text{ preferentially absorbed} \end{array} \right.$

both cause the BH to spin down.

It turns out that angular momentum
 is efficiently radiated away, so that
 a/M tends to decrease (Calculations of
 Page; see the Membrane Paradigm.)

Note also, that because of its
 tendency to spin down, a Kerr BH
 cannot actually be in equilibrium
 with a thermal bath — unless the
 bath is also spinning (with
 "chemical potential" $\mu = \beta \Omega_H$.)

Black Hole Thermodynamics

Naively, a black hole seems to provide a mechanism for violating the second law of Thermodynamics. We can dump some gas into the hole; when it crosses the horizon, we lose access to the gas, and to its entropy.

But the thermal emission from the black hole suggests that black holes do not supersede the second law, but rather require a modification in how it is stated: We must define a new entropy

$$S' = S_{\text{everything else}} + S_{\text{black hole}};$$

This is the quantity that is nondecreasing; the loss of the entropy of the gas is (at least) compensated by the gain of the entropy of the black hole.

Using the calculated temperature $T = \frac{1}{8\pi M}$, we can compute the intrinsic entropy of the black hole, by considering a black hole in equilibrium with a radiation bath. Our calculation showed that a black hole surrounded by radiation at this temperature will emit and absorb at the same rate. (The same transmission probability $1/E^2$ enters into both

processes.) Let us demand that the generalized entropy S' is stationary for this state. (We leave aside, for the moment, the question whether S' is at a local maximum; i.e., whether the equilibrium is stable.) We have

$$\Delta S' = 0 = \left(\frac{\partial S}{\partial E} \right)_{\text{rad}} (\Delta E)_{\text{rad}} + \left(\frac{\partial S}{\partial E} \right)_{\text{BH}} (\Delta E)_{\text{BH}}$$

$$\text{and } (\Delta E)_{\text{rad}} + (\Delta E)_{\text{BH}} = 0 \Rightarrow$$

$$\left(\frac{dS}{dE} \right)_{\text{BH}} = \frac{1}{T_{\text{rad}}} = 8\pi M_{\text{BH}}$$

$$\text{Integrating } dS_{\text{BH}} = 8\pi M_{\text{BH}} dM_{\text{BH}},$$

$$\text{we find } S_{\text{BH}} = 4\pi M_{\text{BH}}^2 + \text{constant}$$

Ignoring the constant (see below), this may be written as

$$S_{\text{BH}} = \frac{1}{4} A, \text{ where } A = \text{area of event horizon.}$$

In this form, the formula generalizes to the Kerr black hole, with $A = 4\pi(r_H^2 + a^2)$.

Putting in numerical constants:

$$S/k_B = \frac{1}{4} \frac{c^3}{G\hbar} A = \frac{1}{4} \frac{A}{L_{\text{Planck}}^2}$$

— area in Planck units

It is appropriate to ignore the additive constant of integration in S , if one believes that the entropy of a Planck size BH should be of order one, and $M \gg M_{\text{Planck}}$.

Why should a black hole have this enormous intrinsic entropy (of order 10^{78} for solar mass black hole)? Perhaps because, once a horizon forms, (almost) all information about how it was made becomes inaccessible, even in principle, to an observer outside the horizon. (Only energy, angular momentum, and charge, if any, can be measured — as for a thermol reservoir when we are completely ignorant of its microscopic state.) In some sense

$$\exp(S_{\text{BH}}/k_B) = \exp\left(\frac{1}{4}A\right)$$

is the number of "microscopic states" accessible to a black hole. It is as though there is a membrane at the horizon, one Planck length deep, with about one bit of information residing in each Planck volume of the membrane.

This idea that the BH has huge intrinsic entropy was anticipated by Bekenstein, before the discovery of the Hawking emission. He was inspired by the theorem that no classical process can reduce the area of the horizon

(a result also due to Hawking) — which bears an obvious resemblance to second law. Bekenstein queried that S_{BH} was of order area in Planck units, but the numerical coefficient could not be determined until $T = (8\pi M)^{-1}$ was computed.

Black Hole in a Radiation Cavity

The equilibrium between BH and a very large radiation bath is actually unstable, because specific of a black hole is negative — its temperature increases as it radiates heat away.

The entropy and energy of a BH at temperature T are

$$S_{BH} = \frac{1}{16\pi G T^2}, \quad E_{BH} = \frac{1}{8\pi G T}$$

— so both decrease as temperature is raised. Contrast this behavior with that of a box filled with a gas of massless quanta with temperature T :

$$E = a V T^4$$

$$a = \frac{\pi^2}{30} (B + \frac{7}{8} F)$$

$$S = \int \frac{dE}{T} = \frac{4}{3} a V T^3$$

B = no. of boson helicity states
 F = no. of fermi helicity states

If we put BH with temperature T in a box of radiation at temperature T , and the box is very large, the equilibrium is unstable:

- If BH accretes radiation, it cools down, accretes more -
- If BH loses mass by emitting, it heats up, emits more -

But if the box is sufficiently small, the equilibrium will be stable. The criterion for local stability is that when energy is transferred from BH to bath, the radiation heats up even more than the BH does. (And when energy is transferred from bath to BH, radiation cools down even more than BH does.)

$$\text{since } T_R \sim E_R^{1/4} \Rightarrow dT_R = \frac{1}{4} T_R \frac{dE_R}{E_R}$$

$$T_{BH} \sim E_{BH}^{-1} \Rightarrow dT_{BH} = -T_{BH} \frac{dE_{BH}}{E_{BH}}$$

The criterion for local stability is

$$E_{\text{radiation}} < \frac{1}{4} M_{\text{black hole}}$$

If this criterion is not met (if the box is too big) the BH will evaporate completely.

Now consider the condition for global stability. That is, given a box with volume V containing energy E , how should E be apportioned

between BH and the radiation gas in order to maximize the total entropy.

Note first of all that it will never pay to have more than one BH. Since $S_{BH} = 4\pi M_{BH}^2$ (for Schwarzschild BH) the entropy of BH's of mass M_1 and M_2 will be increased by the ratio

$$\frac{(M_1 + M_2)^2}{M_1^2 + M_2^2} > 1$$

if the two are merged to form a BH of mass $M_1 + M_2$.

We need to maximize the function

$$S = 4\pi M_{BH}^2 + \frac{4}{3}(\alpha V)^{1/4} E_R^{3/4}$$

where $M_{BH} + E_R = E = \text{constant};$

or, if we rescale

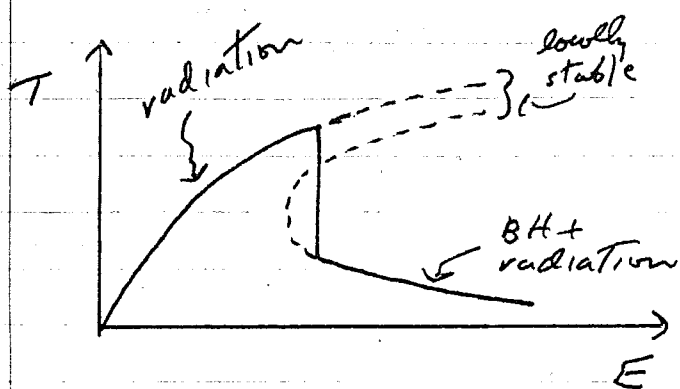
$$S = 4\pi E^2 f(x)$$

where $f(x) = x^2 + c(1-x)^{3/4},$

$$x = \frac{M_{BH}}{E}, \quad c = \frac{1}{3\pi} \left(\frac{\alpha V}{E^5} \right)^{1/4}$$

So we determine x by maximizing $f(x)$ for x in the interval $[0, 1]$.

For C large (big box) the maximum occurs at $x=0$; pure radiation is favored. As more energy is pumped into a cavity of fixed volume V , eventually a BH in equilibrium with radiation becomes locally stable, but the global maximum of the entropy still occurs at $x=0$, until E is somewhat higher.



The temperature T in the box, as a function of the total energy E behaves as shown. The solid line is the temperature of the globally stable configuration, and the dotted line indicates locally stable configurations.

Exercises:

- (i) For a black hole with temperature 300°K , what is the maximum volume V of a cavity filled with gravitons at 300°K such that the black hole is locally stable?

Remark: the cavity can be big, because the energy of a black hole ($E_{\text{BH}} = M_{\text{BH}} c^2$) with $T_{\text{BH}} = 300^\circ\text{K}$ is enormous.