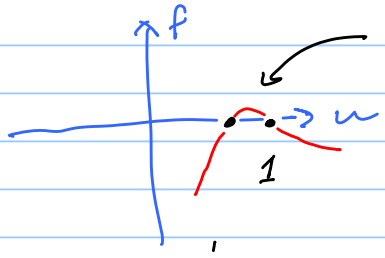
Physics 106a
Problem Set 5 Solutions

1. Sleeping top.

(a) In general, $\dot{u}^2 = f(u) = (1-u^2)(\alpha - \beta u) - (b - a u)^2$.



we are interested in the case where the upper turning point is $u_2 = 1$, and the lower turning point is $u_1 = 1 - \varepsilon_0$, where $\varepsilon_0 \ll 1$.

$f(u_2) = f(1) = 0 \Rightarrow b = a$. Let $u = 1 - \varepsilon$, then

$$f(u) = \varepsilon(2-\varepsilon)[\alpha - \beta(1-\varepsilon)] - a^2\varepsilon^2$$
$$= 2(\alpha-\beta)\varepsilon - (a^2 - 2\beta + \alpha - \beta)\varepsilon^2 - \beta\varepsilon^3.$$

ε is small, so we can neglect the term cubic in ε. Then

$$f(u) = C\varepsilon(\varepsilon_0 - \varepsilon), \text{ where } C = a^2 - 2\beta + (\alpha - \beta)$$
$$\varepsilon_0 = 2(\alpha-\beta)/C$$

$u = \cos\theta \simeq 1 - \frac{1}{2}\theta^2$, for θ small

so $\varepsilon = \frac{1}{2}\theta^2$, and $\dot{u}^2 = (-\theta\dot{\theta})^2$

$$\Rightarrow \dot{u}^2 = \theta^2\dot{\theta}^2 = f(u) = C\tfrac{1}{2}\theta^2\left(\tfrac{1}{2}\theta_0^2 - \tfrac{1}{2}\theta^2\right)$$
$$\Rightarrow \dot{\theta}^2 + \omega^2\theta^2 = \omega^2\theta_0^2, \text{ where } \omega^2 = \tfrac{1}{4}C$$

Furthermore, since ε_0 is small, $\alpha - \beta \ll C$, and we may write $C = a^2 - 2\beta$,

where $a = \dfrac{I_3\omega_s}{I}$ and $\beta = \dfrac{2Mgl}{I}$. Thus $C = \dfrac{I_3^2}{I^2}(\omega_s^2 - \omega_{crit}^2)$

$$\Rightarrow \boxed{\omega = \frac{1}{2}\frac{I_3}{I}(\omega_s^2 - \omega_{crit}^2)^{1/2}}$$

(b) In general, $\dot{\phi} = \dfrac{b - au}{1-u^2} \Rightarrow \dot{\phi} = \dfrac{a(1-u)}{(1-u)(1+u)} = \dfrac{a}{1+u} = \dfrac{a}{2-\varepsilon}$

If ε is small, then when $a = b$

$\dot{\phi} \simeq \frac{1}{2}a = \text{constant} \Rightarrow \Delta\phi = \dot{\phi}\,t_0 = \dot{\phi}(\pi/\omega)$

Therefore, $\Delta\phi = \frac{1}{2}\frac{I_3\omega_s}{I}\left(\frac{2\pi I}{I_3}\right)\left(\omega_s^2 - \omega_{crit}^2\right)^{-\frac{1}{2}}$

$= \pi\left(1 - \omega_{crit}^2/\omega_s^2\right)^{-\frac{1}{2}} \Longrightarrow \pi$

for $\omega_s >> \omega_{crit}$

2. Sliding top

(a) $T = T_{CM} + T_{rot}$

$T_{CM} = \frac{1}{2}M\left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2 + \dot{z}_{cm}^2\right)$ and $z_{CM} = l\cos\theta \Rightarrow$

$\dot{z}_{cm} = -l\dot{\theta}\sin\theta$

$\Rightarrow T_{CM} = \frac{1}{2}M\left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2\right) + \frac{1}{2}Ml^2\sin^2\theta\,\dot{\theta}^2$,

$T_{rot} = \frac{1}{2}I\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2$

$V = Mgz = Mgl\cos\theta$

$L = T_{CM} + T_{rot} - V$

(b) Conserved quantities are

$p_x = M\dot{x}_{cm}$ $p_y = M\dot{y}_{cm}$, $p_\psi = I\dot{\psi} + \dot{\phi}\cos\theta \equiv Ia$

$p_\phi = I\dot{\phi}\sin^2\theta + Ia\cos\theta \equiv Ib$

$E = \frac{1}{2}M\dot{x}_{cm}^2 + \frac{1}{2}M\dot{y}_{cm}^2 + \frac{1}{2}Ml^2\sin^2\theta\,\dot{\theta}^2$

$+ \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}I\frac{(b - a\cos\theta)^2}{\sin^2\theta} + \frac{1}{2}\frac{(Ia)^2}{I_3} + Mgl\cos\theta$

$\Rightarrow \dot{\theta}^2\sin^2\theta + \frac{Ml^2}{I}\dot{\theta}^2\sin^4\theta = -(b - a\cos\theta)^2$

$+ \frac{2}{I}\left(E - \frac{p_x^2 + p_y^2}{2M} - \frac{I^2a^2}{2I_3} - Mgl\cos\theta\right)\sin^2\theta$

$= \sin^2\theta\,(\alpha - \beta\cos\theta) - (b - a\cos\theta)^2$

where $\beta = \frac{2Mgl}{I}$, $\alpha = \frac{2}{I}\left(E - \frac{p_x^2 + p_y^2}{2M} - p_\psi^2/2I_3\right)$

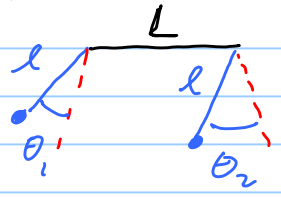
Defining $u = \cos\theta$, $\dot{u}^2\left(I + \gamma(1 - u^2)\right)$ $\left(\gamma = Ml^2/I\right)$

$= (1 - u^2)(\alpha - \beta u) - (b - au)^2$

$\Rightarrow \boxed{\dot{u}^2 = g(u) = \left[(1 - u^2)(\alpha - \beta u) - (b - au)^2\right]/(I + \gamma - \gamma u^2)}$

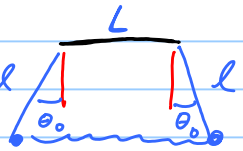
3. Small oscillations

(i) Let θ_1, θ_2 be deviations from equilibrium

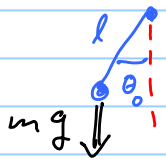


$\Rightarrow KE = \frac{1}{2} m l^2 (\dot\theta_1^2 + \dot\theta_2^2) \Rightarrow T = ml^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

In equilibrium, the spring's length is $L + 2l\sin\theta_0$



$\Rightarrow$ in equilibrium, horizontal force exerted by spring on each mass is $K(L_0 - L - 2l\sin\theta_0)$



The force F exerted on the rod obeys $F\cos\theta_0 = mg$ so the horizontal force on the mass is

$F\sin\theta_0 = mg\tan\theta_0 = K(L_0 - L - 2l\sin\theta_0)$

This eqn determines θ_0.

The gravitational PE is $-mgl\cos(\theta_0+\theta_1) - mgl\cos(\theta_0 - \theta_2)$

Expanding: $\cos(\theta_0 + x) = \cos\theta_0 - x\sin\theta_0 - \frac{1}{2}x^2\cos\theta_0 + \cdots$

We know linear terms cancel in equilibrium. Quadratic term is

$$\text{Grav PE} = \frac{1}{2} mgl\cos\theta_0 (\theta_1^2 + \theta_2^2)$$

$$\text{Spring PE} = \frac{1}{2}K\left[L_0 - L - l\sin(\theta_0+\theta_1) - l\sin(\theta_0 - \theta_2)\right]^2$$

$$= \frac{1}{2}K\left[L_0 - L - 2l\sin\theta_0 - l(\theta_1 - \theta_2)\cos\theta_0 + \frac{1}{2}l(\theta_1^2 + \theta_2^2)\sin\theta_0 + \cdots\right]^2$$

Keeping only quadratic terms,

$$\text{Spring PE} = K(L_0 - L - 2l\sin\theta_0)\frac{1}{2}l(\theta_1^2 + \theta_2^2)\sin\theta_0 + \frac{1}{2}Kl^2\cos^2\theta_0(\theta_1 - \theta_2)^2$$

$$= \frac{1}{2}mgl\tan\theta_0\sin\theta_0(\theta_1^2 + \theta_2^2) + \frac{1}{2}Kl^2\cos^2\theta_0(\theta_1 - \theta_2)^2$$

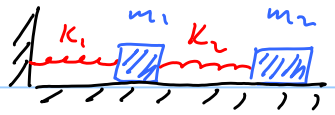
(where we have substituted equilibrium value of $L_0 - L - 2l\sin\theta_0$)

Thus $V = ml^2 \begin{pmatrix} A & B \\ B & A \end{pmatrix}$

$$A = \frac{g}{l}\frac{\sin^2\theta_0}{\cos\theta_0} + \frac{K}{m}\cos^2\theta_0$$

$$B = -\frac{K}{m}\cos^2\theta_0$$

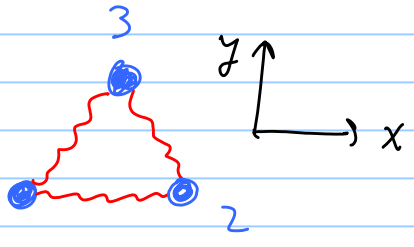
(ii)



$$KE = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \Rightarrow T = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

$$PE = \frac{1}{2} K_1 x_1^2 + \frac{1}{2} K_2 (x_2 - x_1)^2 \Rightarrow V = \begin{pmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{pmatrix}$$

(iii)



Define:

$$\eta = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix}$$

x_1, y_1: Deviation of mass #1 from equilibrium.
x_2, y_2: Deviation of mass 2.
x_3, y_3: Deviation of mass 3.

$$KE = \frac{1}{2} m \left(\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2} + \dot{\vec{r}}_3^{\,2} \right)$$

$\Rightarrow T = m \mathbb{1}$ ← 6×6 identity matrix

(Equilibrium config. is equilateral triangle.)

To find PE for small displacement from equilibrium, note that only displacement parallel to spring stretches the spring. The change in the length of the spring connecting ① and ② is component of $\vec{r}_2 - \vec{r}_1$ along $(1, 0)$, The change in length of spring connecting ① and ③ is component of $\vec{r}_3 - \vec{r}_1$ along $\left(\frac{1}{2}, \sqrt{3}/2\right)$, and the change in length of spring connecting ② and ③ is component of $\vec{r}_3 - \vec{r}_2$ along $\left(-\frac{1}{2}, \sqrt{3}/2\right)$. Therefore,

$$V = \frac{1}{2} K (x_1 - x_2)^2 + \frac{1}{2} K \left[\frac{1}{2}(x_1 - x_3) + \frac{\sqrt{3}}{2}(y_1 - y_3) \right]^2 + \frac{1}{2} K \left[-\frac{1}{2}(x_2 - x_3) + \sqrt{3}/2 (y_2 - y_3) \right]^2 .$$

So, for example, the x_1^2 term is $\frac{1}{2} k \left(1 + \frac{1}{4}\right) x_1^2 = \frac{1}{2} k \left(5/4 \, x_1^2 \right)$,

The y_1^2 term is $\frac{1}{2}K\left(\frac{3}{4}y_1^2\right)$

The x_1y_1 term is $K\left(\frac{1}{2}\cdot\frac{\sqrt{3}}{2}x_1y_1\right) = K\left(\frac{\sqrt{3}}{4}x_1y_1\right)$,

etc., etc.

Assembling all terms, we find the 6×6 matrix:

$$\tilde{\tilde{V}}/K = \begin{bmatrix} 5/4 & \sqrt{3}/4 & -1 & 0 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & 0 & 0 & -\sqrt{3}/4 & -3/4 \\ -1 & 0 & 5/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ 0 & 0 & -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & 1/2 & 0 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & -3/4 & 0 & 3/2 \end{bmatrix}$$