Physics 106a
Problem Set 6 Solution

$$T = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad V = \begin{pmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{pmatrix}$$

(1) The normal modes satisfy $0 = \det(-\omega^2 T + V) = \det\begin{pmatrix} -\omega^2 m_1 + K_1 + K_2 & -K_2 \\ -K_2 & -\omega^2 m_2 + K_2 \end{pmatrix}$

$$\Rightarrow 0 = m_1 m_2 \omega^4 - ((K_1+K_2)m_2 + K_2 m_1)\omega^2 + K_1 K_2$$

$$\Rightarrow \boxed{\omega^2_{1,2} = \frac{1}{2m_1m_2}\left[(K_1+K_2)m_2 + K_2 m_1 \pm \sqrt{(m_2K_1 + m_2K_2 + m_1K_2)^2 - 4K_1K_2m_1m_2}\right]}$$

The normal vectors $X^a = \begin{pmatrix} u^a \\ v^a \end{pmatrix}$ satisfy

$$(-\omega_a^2 T + V)X^a = 0 \Rightarrow (-m_2\omega_a^2 + K_2)v^a - K_2 u^a = 0 \Rightarrow$$

$$u^a = \left(1 - \frac{m_2}{K_2}\omega_a^2\right)v^a$$

so $X^a = N^a \begin{pmatrix} 1 - \frac{m_2}{K_2}\omega_a^2 \\ 1 \end{pmatrix}$

where N^a is the normalization factor determined by

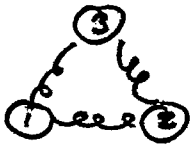
$$\tilde{X}^a T X^a = (N^a)^2\left[\left(1 - \frac{m_2}{K_2}\omega_a^2\right)^2 m_1 + m_2\right] = 1$$

$$\boxed{X^a = \frac{1}{\sqrt{\left(1 - \frac{m_2}{K_2}\omega_a^2\right)^2 m_1 + m_2}} \begin{pmatrix} 1 - \frac{m_2}{K_2}\omega_a^2 \\ 1 \end{pmatrix}}$$

The normal coordinates are

$$\boxed{\xi^a = \tilde{X}^a T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{\left(1 - \frac{m_2}{K_2}\omega_a^2\right)^2 m_1 + m_2}}\left(m_1\left(1 - \frac{m_2}{K_2}\omega_a^2\right)x_1 + m_2 x_2\right)}$$

(2)

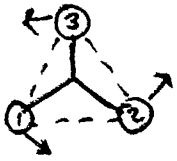


$$T = m\mathbb{1} \qquad V = K \begin{bmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & \\ 5/4 & \sqrt{3}/4 & -1 & 0 & -1/4 & -\sqrt{3}/4 & y_1 \\ \sqrt{3}/4 & 3/4 & 0 & 0 & -\sqrt{3}/4 & -3/4 & y_2 \\ -1 & 0 & 5/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & x_2 \\ 0 & 0 & -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 & y_2 \\ -1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & 1/2 & 0 & x_3 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & -3/4 & 0 & 3/2 & y_3 \end{bmatrix}$$

(The expression for V given last week.)

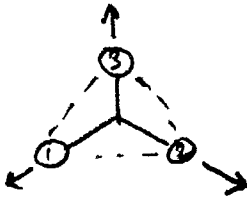
Two of the "normal modes" are translations in the x and y directions

$$X^{(1)} = \frac{1}{\sqrt{3m}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \omega_1^2 = 0 \qquad X^{(2)} = \frac{1}{\sqrt{3m}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \omega_2^2 = 0$$



Another mode with zero frequency is rotation about the center of mass

$$X^{(3)} = \frac{1}{\sqrt{3m}} \begin{pmatrix} \frac{1}{2} \\ -\sqrt{3}/2 \\ \frac{1}{2} \\ \sqrt{3}/2 \\ -1 \\ 0 \end{pmatrix}, \quad \omega_3^2 = 0$$

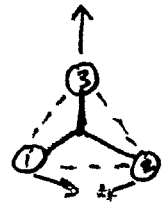


It is easy to guess that there is a completely symmetric "breathing" mode:

$$X^{(4)} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -\sqrt{3}/2 \\ -\frac{1}{2} \\ \sqrt{3}/2 \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

$$\upsilon X^{(4)} = 3KX^{(4)} = \omega_4^2 \mathcal{L} X^{(4)}$$

$$\Rightarrow \boxed{\omega_4^2 = \frac{3K}{m}}$$



There is another mode symmetric about the y axis:

$$X^{(5)} = \frac{1}{\sqrt{3m}} \begin{pmatrix} a \\ b \\ -a \\ b \\ 0 \\ 1 \end{pmatrix}$$

To find a and b, we can use orthogonality.

$$0 = \tilde{X}^{(2)} X^{(5)} = 2b + 1 \Rightarrow b = -\tfrac{1}{2}$$

$$0 = \tilde{X}^{(4)} X^{(5)} = -\sqrt{3}a - b + 1 \Rightarrow a = \sqrt{3}/2$$

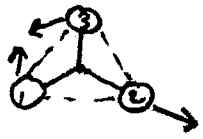
$$X^{(5)} = \frac{1}{\sqrt{3m}} \begin{pmatrix} \sqrt{3}/2 \\ -\frac{1}{2} \\ -\sqrt{3}/2 \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

$$\upsilon X^{(5)} = \tfrac{3}{2} K X^{(5)} \Rightarrow \boxed{\omega_5^2 = \frac{3K}{2m}}$$

Because of the manifest symmetry under 120° rotations in the x-y plane,

$$X_1^{(6)} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -\sqrt{3}/2 \\ -\frac{1}{2} \\ 0 \\ 1 \\ \sqrt{3}/2 \\ -\frac{1}{2} \end{pmatrix}$$

and



$$X_2^{(6)} = \frac{1}{\sqrt{3m}} \begin{pmatrix} 0 \\ 1 \\ \sqrt{3}/2 \\ -\frac{1}{2} \\ -\sqrt{3}/2 \\ -\frac{1}{2} \end{pmatrix}$$

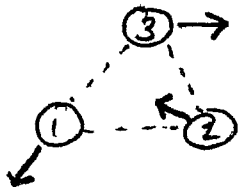
are also normal modes with frequency $\omega_6^2 = 3K/2m$. $X^{(5)}$, $X_1^{(6)}$, and $X_2^{(6)}$ are not independent.

$$X^{(5)} + X_1^{(6)} + X_2^{(6)} = 0$$

We want to find a linear combination of $X_1^{(6)}$ and $X_2^{(6)}$ which is orthogonal to $X^{(5)}$. Since $\tilde{X}^{(5)} X_1^{(6)} = -1/2 = \tilde{X}^{(5)} X_2^{(6)} = \tilde{X}_1^{(6)} X_2^{(6)}$, we choose,

$$X^{(6)} = \frac{1}{\sqrt{3}} \left(X_1^{(6)} - X_2^{(6)} \right) \Rightarrow$$

$$\boxed{X^{(6)} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -\frac{1}{2} \\ -\sqrt{3}/2 \\ -\frac{1}{2} \\ \sqrt{3}/2 \\ 1 \\ 0 \end{pmatrix}, \quad \omega_6^2 = \frac{3K}{2m}}$$



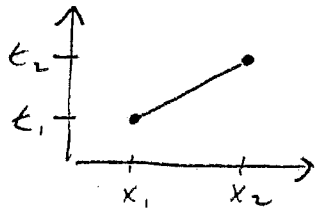
The normal coordinates are $\xi^{(a)} = m \tilde{\chi}^{(a)} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix}$

(3)

$$L = \tfrac{1}{2} m \dot{x}^2$$

Eq of motion is $m\ddot{x} = 0$

or $v = \dot{x} = \text{constant}$



$$\Rightarrow \quad x = x_1 + v(t - t_1)$$

Since $x(t_2) = x_2$, we have $v = \dfrac{x_2 - x_1}{t_2 - t_1}$

Action is ... $S = \displaystyle\int_{t_1}^{t_2} \tfrac{1}{2} m v^2 = \tfrac{1}{2} m v^2 (t_2 - t_1)$

or $$\boxed{S = \tfrac{1}{2} m \frac{(x_2 - x_1)^2}{t_2 - t_1}}$$

$$\frac{\partial S}{\partial x_2} = m \frac{x_2 - x_1}{t_2 - t_1} = mv = p = p(t_2)$$

(of course, p is a constant for free particle.)

$$\frac{\partial S}{\partial t_2} = -\tfrac{1}{2} m \frac{(x_2 - x_1)^2}{(t_2 - t_1)^2} = -\tfrac{1}{2} m v^2 = -H = -H(t_2)$$

(4)

$$H = \sum_a \frac{1}{2m_a} \vec{P}_a^{\,2} + \sum_{a<b} V(\vec{x}_a - \vec{x}_b) \equiv T + V$$

Virial Theorem $\Rightarrow$ $\langle T \rangle = \left\langle \tfrac{1}{2} q_i \dfrac{\partial H}{\partial q_i} \right\rangle$

$$= \left\langle \tfrac{1}{2} x_{ai} \frac{\partial}{\partial x_{ai}} V \right\rangle$$

(x_{ai} denotes the ith component of the position of particle a)

Note that $$x_{ci} \frac{\partial}{\partial x_{ci}} V(\vec{x}_a - \vec{x}_b) = \left(x_{ai} \frac{\partial}{\partial x_{ai}} + x_{bi} \frac{\partial}{\partial x_{bi}} \right) V(\vec{x}_a - \vec{x}_b)$$

$$= (\vec{x}_a - \vec{x}_b) \cdot \vec{\nabla} V(\vec{x}_a - \vec{x}_b)$$

So $$\langle T \rangle = \left\langle \tfrac{1}{2} \sum_{a<b} (\vec{x}_a - \vec{x}_b) \cdot \vec{\nabla} V(\vec{x}_a - \vec{x}_b) \right\rangle$$

Gravitational interaction: $V(\vec{r}) = \dfrac{1}{r} = \dfrac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$

$$\Rightarrow \quad \vec{x} \cdot \vec{\nabla} V = \sum_i x_i \frac{-x_i}{r^{3/2}} = -\frac{1}{r} = -V(\vec{r})$$

Thus $$\boxed{\langle T \rangle = -\tfrac{1}{2} \langle V \rangle}$$