Lecture 14
18 May 2026
Ph/CS 219

Cluster states and the one-way QC

Last Time: Pauli-based computation. Universal QC via nondestructive measurements (chosen adaptively) of high-weight Pauli product operators, where all inputs are $|T\rangle$ states. This is useful for FTQC: we can do reliable universal QC if we can prepare clean $|T\rangle$ states and perform FT measurements. We'll discuss how to build FT architectures based on this idea, but not until the next lecture.

First, though, we'll discuss a related type of measurement-based QC (MBQC), in which the computation proceeds via adaptive single-qubit measurements acting on an entangled many-qubit resource state. This is conceptually interesting, and also has applications to FT – for example in photonic QC, where each photon is measured shortly after entanglement with other photons is established. As we'll discuss, it is also an instructive example of a symmetry-protected topological phase (SPT phase), an important idea in quantum condensed matter physics.

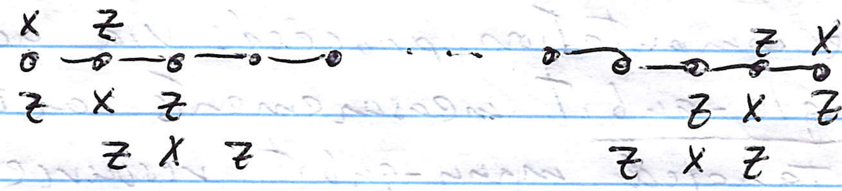
This scheme is sometimes called the

"one-way quantum computer" because the entangled resource is irreversibly consumed by destructive measurements as the computation proceeds.

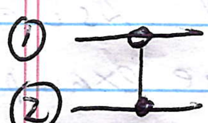
For universal QC we'll need a 2D resource state. But first consider 1D as a warm-up example.

The 1D cluster state (also called a graph state) is a stabilizer state, the simultaneous +1 eigenstate of Pauli operators

$$X_1 Z_2, \quad Z_{i-1} X_i Z_{i+1} \ (i=2,3,\dots,n-1), \quad Z_{n-1} X_n$$



These commute: 2 or 0 collisions of X with Z. The state is easy to prepare. Start with $|+\rangle^{\otimes n}$ and apply CZ to all neighboring pairs



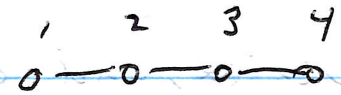
CZ is symmetric under 1 $\leftrightarrow$ 2.

It commutes with Z_1, Z_2

Takes $X_1 = \pm 1$ eigenstates to $X_1 Z_2 = \pm 1$ eigenstates

$$|\pm\rangle|\psi\rangle \to \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle \pm |1\rangle Z|\psi\rangle\right)$$

Hence $CZ: X_1 \to X_1 Z_2, \ X_2 \to Z_1 X_2, \ Z_1 \to Z_1, \ Z_2 \to Z_2$



$CZ(1,2)\,CZ(2,3)\,CZ(3,4)$

$X_1 \to X_1 Z_2$
$X_2 \to Z_1 X_2 Z_3$
$X_3 \to Z_2 X_3 Z_4$
$X_4 \to Z_3 X_4$

Hence $|+\rangle^{\otimes 4} \to$ cluster state.

All CZ gates commute and can be done simultaneously in depth = 1. And this works for <u>any</u> graph

$$X_i \xrightarrow{CZ_s} X_i \bigotimes_{j \in \partial i} Z_j$$

where ∂i means the set of vertices connected to i by edges.

The cluster state is a unique stabilizer state for any graph (number of independent stab generators = number of qubits). Obtain encoded information by removing one or more stabilizers. E.g. in 1D remove the stab gen at left boundary $X_1 Z_2$. Now there is encoded qubit with logical operators

$$\bar{Z} = Z_1 \quad \bar{X} = X_1 Z_2$$

these anticommute with each other, commute with stabilizer.

Consider measuring the qubits one at a time, starting at left boundary.

Measure $Z_1 = \bar{Z}$. A Z measurement of a logical qubit

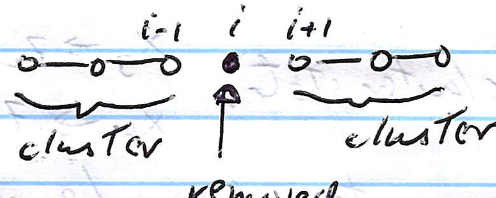
Measure Z_i in chain interior. Outcome $Z_i = (-1)^a$

Then: $Z_{i-2} X_{i-1} Z_i \to (-1)^a Z_{i-2} X_{i-1}$. New stab

$Z_i X_{i+1} Z_{i+2} \to (-1)^a X_{i+1} Z_{i+2}$. New stab

$Z_{i-1} X_i Z_{i+1} \to Z_i = (-1)^a$ New stab

This removes the ith vertex, and cuts the chain into two disjoint cluster states



(up to Pauli correction Z_{i-1}^a, Z_{i+1}^a).

Measure $X_1 = (-1)^a$, removes left boundary site

$$\bar{X} = X_1 Z_2 \to (-1)^a Z_2 = (-1)^a \bar{Z}'$$ the logical Z of shortened chain.

$$\bar{Z} = Z_1 \equiv X_2 Z_3 = \bar{X}'$$ the logical X of shortened chain.

This is a logical Hadamard, up to Pauli correction X^a.

We can imagine preparing cluster state one CZ at a time, in each step followed by X measurement.

① $|\psi\rangle$ —●— $\boxed{M_X} = (-1)^a$ stab $X_2 \xrightarrow{CZ} Z_1 X_2$ new stab.

② $|+\rangle$ —●— $X^a H|\psi\rangle$ then $Z_1 \equiv X_2$

This is H up to Pauli. $X_1 \to X_1 Z_2 \to (-1)^a Z_2$

$H: X \to Z, Z \to X$, $X^a: Z \to (-1)^a Z$

Now, apply $CZ(2,3)$ to $X^a H|\psi\rangle_2 \otimes |+\rangle_3$ and measure $X_2 = (-1)^b$

Propagation: $|\psi\rangle \to X^b H X^a H |\psi\rangle = X^b Z^a |\psi\rangle$.

Measuring the two sites at left boundary preserves $|\psi\rangle$ up to Pauli frame update determined by X measurements.

To generate single-qubit Clifford, we need S in addition to H.

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}: \quad Z \to Z, \quad X \to -Y$$

Alternatively, HS: $X \xrightarrow{S} -Y \xrightarrow{H} Y$

$$Z \xrightarrow{S} Z \xrightarrow{H} X$$

obtain this by measuring Y

$$① \; |\psi\rangle \text{ —•— } \boxed{M_Y} = (-1)^a$$
$$② \; |+\rangle \text{ —•— } X^a HS|\psi\rangle$$

Stab: $X_2 \to Z_1 X_2$

Then $Z_1 \equiv X_2$,

$$X_1 \to X_1 Z_2 \equiv Y_1 Y_2 \to (-1)^a Y_2 \overset{X_2^a}{\Rightarrow} Y_2.$$

For universal 1-Qubit computation we also need the T gate. From the above, applying $CZ(1,2)$ and M_{X_1} to the input state $T|\psi\rangle_1$, we have

$$① \; |\psi\rangle - \boxed{T} \text{ —•— } \boxed{M_X} = (-1)^a$$
$$② \; |+\rangle \text{ —•— } X^a H T|\psi\rangle$$

And T gate commutes through CZ.

$- \boxed{T} - \boxed{M_X} = (-1)^a$ is measurement in basis

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \qquad T|\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm e^{i\pi/4}|1\rangle\right)$$

we apply the above circuit to $X^b|\psi\rangle$ where b is determined by how many times we have obtained outcome (-1) in preceding X and Y measurements (whether number is even or odd).

outcome is $X^a H T X^b |\psi\rangle$, and we want to pull X^b out in front to update Pauli frame. But (up to a phase) $TX = XT^{-1}$!

$$TX = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ e^{i\pi/4} & 0 \end{pmatrix}$$

$$XT^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} = \begin{pmatrix} 0 & e^{-i\pi/4} \\ 1 & 0 \end{pmatrix} = e^{-i\pi/4} TX$$

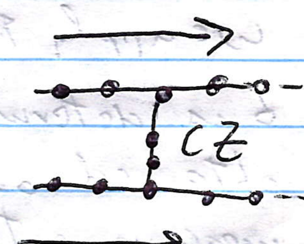
Hence $X^a H T X^b |\psi\rangle = X^a Z^b H T^{-1} |\psi\rangle$ for $b=+1$

$= X^a H T |\psi\rangle$ for $b=0$

If we want to apply T, then measure in basis $T|\pm\rangle$ for $b=0$ and in basis $T^{-1}|\pm\rangle$ for $b=1$. The measurement basis is chosen <u>adaptively</u>.

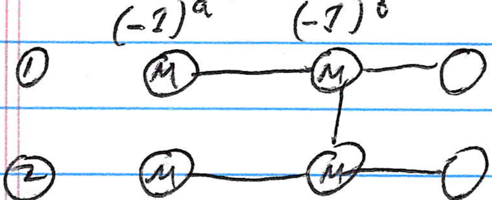
We have generators for 1Q Clifford. For universality we need CNOT or CZ. For this:

1. Build 2D cluster state on square lattice.
2. Carve out 1D wires by removing sites from lattice via Z measurement (or we could have created a graph state with the desired topology to begin with).
3. X measurement propagates qubits along wires. CZ gate (up to conjugation by H) as shown:



④ Perform 1Q Clifford + T via adaptive measurement as described above.

We need to understand how interaction between wires results in CZ gate. We follow how logical operators propagate when X is measured



Multiply by elements of stabilizer so that support of logicals on measured sites is X or I

$$Z_1 = \begin{matrix} Z\,I\,I \\ I\,I\,I \end{matrix} \equiv \begin{matrix} I\,X\,Z \\ I\,Z\,I \end{matrix} \equiv \begin{matrix} I\,X\,Z\,I \\ I\,I\,X\,Z \end{matrix} \rightarrow (-1)^b \begin{matrix} Z\,I \\ X\,Z \end{matrix} = (-1)^b Z_1 X_2$$

Measure X

$$X_1 = \begin{matrix} X\,Z\,I \\ I\,I\,I \end{matrix} \equiv \begin{matrix} X\,I\,X\,Z \\ I\,I\,I\,I \end{matrix} \rightarrow (-1)^a \begin{matrix} X\,Z \\ I\,I \end{matrix} = (-1)^a X_1$$

Measure X

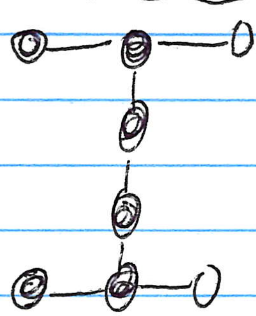
Remove phases with $Z_1^a Z_2^b$

By symmetry: $Z_2 \rightarrow Z_2 X_1$, $X_2 \rightarrow X_2$, up to Pauli.

This is Hadamard-rotated CZ gates

$$\left.\begin{aligned} X_1 &\xrightarrow{H} Z_1 \longrightarrow Z_1 X_2 \xrightarrow{H} X_1 Z_2 \\ Z_1 &\xrightarrow{H} X_1 \longrightarrow X_1 \xrightarrow{H} Z_1 \end{aligned}\right]$$

and similarly, $X_2 \rightarrow Z_1 X_2$, $Z_2 \rightarrow Z_2$.



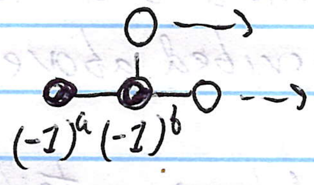
Measure colored sites:

$$Z_1 = \begin{matrix} Z\,I\,I \\ I \\ I \\ I\,I\,I \end{matrix} \equiv \begin{matrix} I\,X\,Z \\ Z \\ I \\ I\,I\,I \end{matrix} \equiv \begin{matrix} I\,X\,Z \\ I \\ X \\ I\,Z\,I \end{matrix} \equiv \begin{matrix} I\,X\,Z\,I \\ I \\ X \\ I\,I\,X\,Z \end{matrix} \rightarrow Z_1 X_2$$

Measure X

up to Pauli.

Alternative formulation:



Stabilizer is $\begin{array}{ccc} & Z & \\ Z & X & Z \end{array}$ $\quad \begin{array}{ccc} & I & \\ I & Z & X \end{array}$

Logical $\bar{Z}_1 = \begin{array}{ccc} & Z & \\ I & I & I \end{array}$ $\quad \bar{Z}_2 = \begin{array}{ccc} & I & \\ Z & I & I \end{array}$

$\bar{X}_1 = \begin{array}{ccc} & X & \\ I & Z & I \end{array}$ $\quad \bar{X}_2 = \begin{array}{ccc} & I & \\ X & Z & I \end{array}$

Now multiply logicals by stabilizers to obtain equivalent logicals

$\bar{Z}_2 \equiv \begin{array}{ccc} & Z & \\ I & X & Z \end{array}$ $\quad \bar{X}_1 \equiv \begin{array}{ccc} & X & \\ I & I & X \end{array}$ $\quad \bar{X}_2 = \begin{array}{ccc} & I & \\ X & I & X \end{array}$

$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$

$(-1)^b \begin{array}{ccc} & Z & \\ \circ & \circ & Z \end{array}$ $\quad \begin{array}{ccc} & X & \\ \circ & \circ & X \end{array}$ $\quad (-1)^a \begin{array}{ccc} & I & \\ \circ & \circ & X \end{array}$

Thus $Z_1 \to Z_1 \qquad X_1 \to X_1 X_2$

$Z_2 \to (-1)^b Z_1 Z_2 \qquad X_2 \to (-1)^a X_2$

This is CNOT; phase can be removed by $X_2^b Z_1^a Z_2^a$.

We have seen how to perform universal QC with adaptive single-qubit measurements acting on a 2D cluster state. Furthermore, we can prepare the cluster state "on the fly" using only CZ gates.