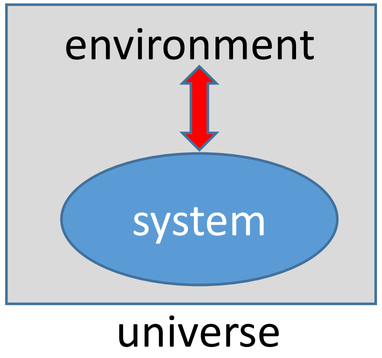
Ph/CS 219A
Quantum Computation

Lecture 4. Quantum Channels



This is the third of several lectures in which we will develop the theory of *open quantum systems*.

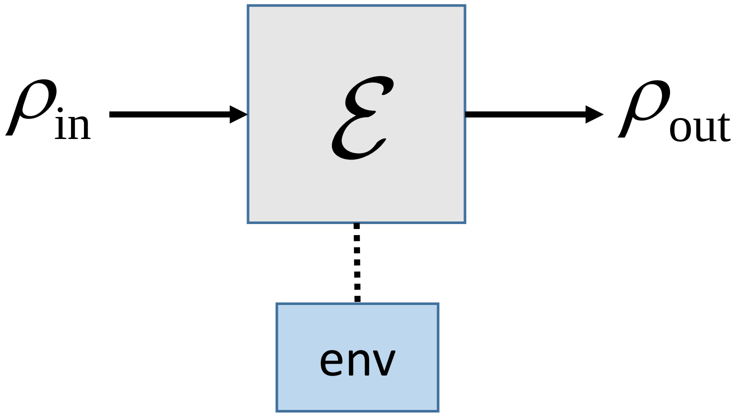
We say that a system is "open" if it can exchange energy and information with its environment.

So far we talked about states of an open system, described by density operators (states which are not necessarily pure), and also about generalized measurements (which are not necessarily orthogonal).

Today we will talk about how the quantum state of an open system can evolve. This evolution is not necessarily unitary.

See Chapter 3 of the Lecture Notes. Note that Homework #1 has been posted, due Friday October 16

Quantum Channels



$$\rho_{\text{out}} = \mathcal{E}(\rho_{\text{in}}) = \sum_a M_a \rho_{\text{in}} M_a^\dagger, \quad \sum_a M_a^\dagger M_a = I.$$

A quantum channel maps density operators to density operators. A special case is a unitary map, which maps a pure quantum state to a pure quantum state.

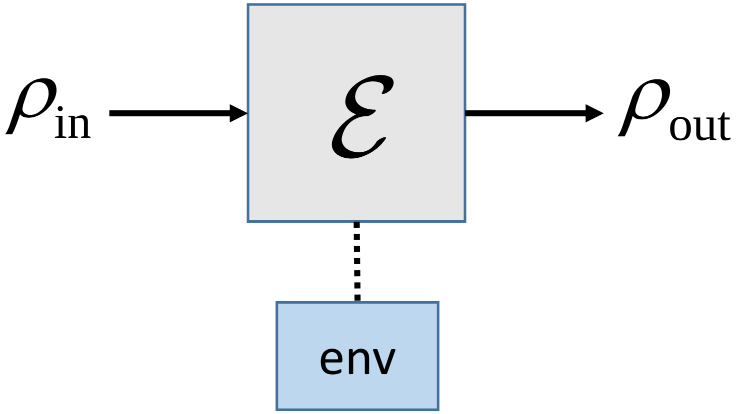
$$\mathcal{E}_U(\rho) = U \rho U^\dagger$$

But a general quantum channel can map pure states to mixed states.

Three properties: A quantum channel is a linear map which preserves:

1) Hermiticity
2) Positivity
3) Trace

Quantum Channels



$$\rho_{\text{out}} = \mathcal{E}(\rho_{\text{in}}) = \sum_a M_a \rho_{\text{in}} M_a^\dagger, \quad \sum_a M_a^\dagger M_a = I.$$

Semigroup property: We can compose two quantum channels to get another quantum channel

$$\rho \underset{\{M_a\}}{\rightarrow} \mathcal{E}_1(\rho) \underset{\{N_\mu\}}{\rightarrow} \mathcal{E}_2(\mathcal{E}_1(\rho)) = \underset{\{N_\mu M_a\}}{\mathcal{E}_2 \circ \mathcal{E}_1(\rho)} \qquad \sum_{a,\mu} M_a^\dagger N_\mu^\dagger N_\mu M_a$$

When is a quantum channel invertible by another quantum channel?

$$\rho_{\text{in}} \underset{\mathcal{E}_1}{\rightarrow} \rho_{\text{out}} \underset{\mathcal{E}_2}{\rightarrow} \rho_{\text{in}}$$

Quantum Channels

Invertible?

$$\mathcal{E}_2 \circ \mathcal{E}_1(|\psi\rangle\langle\psi|) = \sum_{\mu,a} N_\mu M_a |\psi\rangle\langle\psi| M_a^\dagger N_\mu^\dagger = |\psi\rangle\langle\psi| \quad \Rightarrow N_\mu M_a = \lambda_{\mu a} I$$

Completeness:

$$M_b^\dagger M_a = M_b^\dagger \left(\sum_\mu N_\mu^\dagger N_\mu \right) M_a = \sum_\mu \lambda_{\mu b}^* \lambda_{\mu a} I \equiv \beta_{ba} I.$$

Polar decomposition: $$M_a = U_a \sqrt{M_a^\dagger M_a} = \sqrt{\beta_{aa}}\, U_a$$

$$M_b^\dagger M_a = \sqrt{\beta_{aa}\beta_{bb}}\, U_b^\dagger U_a = \beta_{ba} I \quad \Rightarrow \quad U_a = \frac{\beta_{ba}}{\sqrt{\beta_{aa}\beta_{bb}}} U_b$$

All the Kraus operators are proportional to one particular unitary --- an invertible map is unitary! Decoherence (evolution of a pure state to a mixed state) is irreversible. When information leaks to the environment, we can't recover it.

(This argument applies for a map that takes system A to itself or to another system with the same dimension; in that case each Kraus operator is a square matrix, which has a polar decomposition.)

Quantum Operations

After a unitary acts on AB, and B is measured, we might retain some of the information about the measurement outcome, and discard the rest.

$$\sum_{a,\mu} M_{a\mu}^\dagger M_{a\mu} = I$$

$$\mathrm{Prob}(a,\mu) = \mathrm{tr} M_{a\mu}\rho M_{a\mu}{}^\dagger \Rightarrow \mathrm{Prob}(a) = \sum_\mu \mathrm{Prob}(a,\mu)$$

$$\mathcal{E}_a(\rho) \equiv \sum_\mu M_{a\mu}\rho M_{a\mu}^\dagger$$

Post-measurement state:

$$\frac{\mathcal{E}_a(\rho)}{\mathrm{Prob}(a)} = \frac{\mathcal{E}_a(\rho)}{\mathrm{tr}\mathcal{E}_a(\rho)}$$

Sometimes we work with subnormalized states, knowing that we can always divide by the trace to restore the proper normalization. That can be convenient because the map is linear apart from the renormalization.

$$\rho \mapsto \frac{\mathcal{E}_{a_n} \circ \mathcal{E}_{a_{n-1}} \circ \cdots \circ \mathcal{E}_{a_2} \circ \mathcal{E}_{a_1}(\rho)}{\mathrm{tr}\,\mathcal{E}_{a_n} \circ \mathcal{E}_{a_{n-1}} \circ \cdots \circ \mathcal{E}_{a_2} \circ \mathcal{E}_{a_1}(\rho)}$$

Why are Quantum Operations Linear?

Linearity is needed for the ensemble interpretation of the density operator to be self-consistent.

Initial state: $\rho = \sum_i p_i \rho_i, \quad (\rho_i$ prepared with probability p_i).

If ρ_i prepared, measurement outcome a occurs with probability $p(a \mid i)$.

Then *a posteriori*, the state i was prepared with probability $p(i \mid a)$.

Post-measurement state is: $\sum_i p(i \mid a) \dfrac{\mathcal{E}_a(\rho_i)}{p(a \mid i)} \quad = \quad \dfrac{\mathcal{E}_a\left(\sum_i p_i \rho_i\right)}{p_a}$

Bayes Rule : $p(i \mid a) p_a = p(a,i) = p(a \mid i) p_i \Rightarrow \dfrac{p(i \mid a)}{p(a \mid i)} = \dfrac{p_i}{p_a}$

$$\Rightarrow \sum_i p_i \mathcal{E}_a(\rho_i) = \mathcal{E}_a\left(\sum_i p_i \rho_i\right)$$

Complete Positivity

Definition: $\mathcal{E}$ is completely positive on A if $\mathcal{E}_A \otimes I_B$ is positive acting on AB (for any B).

We want a physical operation to map density operators to density operators even if it acts on only part A of a larger system AB (where A and B might be entangled).

And not all positive operators are completely positive. Example: the *transpose* operator:

$$T : |i\rangle\langle j| \mapsto |j\rangle\langle i| \quad \Rightarrow \quad T : \rho \to \rho^T$$

Evidently positive:

$$|\psi\rangle = \sum_i \psi_i |i\rangle, \quad |\psi^*\rangle = \sum_i \psi_i^* |i\rangle \quad \Rightarrow \quad \langle \psi | \rho^T | \psi \rangle = \sum_{i,j} \psi_j^* \left(\rho^T\right)_{ji} \psi_i = \sum_{i,j} \psi_i \left(\rho\right)_{ij} \psi_j^* = \langle \psi^* | \rho | \psi^* \rangle \geq 0$$

Consider (*unconventionally normalized* state): $\quad |\tilde{\Phi}\rangle_{AB} = \sum_{i=0}^{d-1} |i\rangle_A \otimes |i\rangle_B \equiv \sum_i |i,i\rangle$

$$T \otimes I : |\tilde{\Phi}\rangle\langle\tilde{\Phi}| = \sum_{i,j} |i\rangle\langle j| \otimes |i\rangle\langle j| \mapsto \sum_{i,j} |j\rangle\langle i| \otimes |i\rangle\langle j| \equiv \sum_{i,j} |j,i\rangle\langle i,j| \quad = \quad \mathrm{SWAP}$$ which is *not* positive.

$$\mathrm{SWAP} : |\psi\rangle_A \otimes |\varphi\rangle_B = \sum_{i,j} \psi_i \varphi_j |i,j\rangle \quad \mapsto \quad \sum_{i,j} \varphi_j \psi_i |j,i\rangle = |\varphi\rangle_A \otimes |\psi\rangle_B$$

Complete Positivity

$$\mathrm{SWAP}: \quad |\psi\rangle_A \otimes |\varphi\rangle_B = \sum_{i,j} \psi_i \varphi_j \, |i, j\rangle \mapsto \sum_{i,j} \varphi_j \psi_i \, | j, i\rangle = |\varphi\rangle_A \otimes |\psi\rangle_B$$

SWAP has eigenvalue +1 acting on symmetric states, and eigenvalue -1 acting on antisymmetric states.
Example:

$$\mathrm{SWAP}: \quad |01\rangle - |10\rangle \to -\left(|01\rangle - |10\rangle\right)$$

Any physically realizable operation should be a complete positive linear map of density operators to density operators.

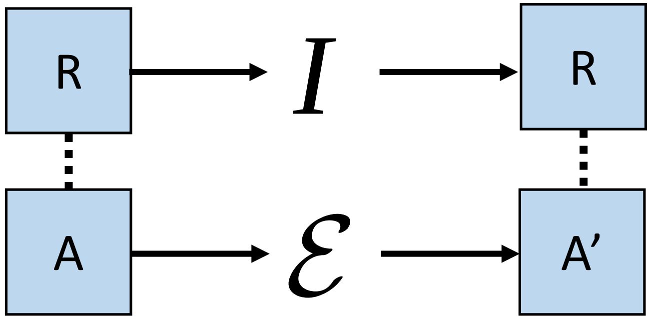
In fact the most general such maps are quantum operations. This means we can always realize the map from A to A' by extending A to AB, performing a unitary operation on AB, and then performing an orthogonal measurement on B.

If there is no postselection on a particular measurement outcome, then the map is a channel, which is also trace preserving; measurements, on the other hand, can *decrease* the trace, in which case the post-measurement state should be renormalized.

To prove this statement (completely positive linear maps are quantum operations), we use the concept: *channel-state duality*.

Channel-State Duality (Choi-Jamiolkowski Isomorphism)

$$|\tilde{\Phi}\rangle_{RA} = \sum_{i=0}^{d-1} |i\rangle_R \otimes |i\rangle_A$$



$$\sum_a \left(|\tilde{\Psi}_a\rangle\langle\tilde{\Psi}_a| \right)_{RA'}$$

Normalization depends on *a*.

"Relative state method":
$$|\varphi\rangle_A = \sum_i \varphi_i |i\rangle_A = \sum_i \varphi_i \left({}_R\langle i|\tilde{\Phi}\rangle_{RA} \right) = {}_R\langle \varphi^* |\tilde{\Phi}\rangle_{RA}$$

$$\Rightarrow \mathcal{E}\left(\left(|\varphi\rangle\langle\varphi| \right)_A \right) = \sum_a \left(\langle \varphi^* |\tilde{\Psi}_a\rangle\langle\tilde{\Psi}_a|\varphi^*\rangle \right)_{A'}.$$

Define a linear operator: $M_a|\varphi\rangle_A = {}_R\langle\varphi^*|\tilde{\Psi}_a\rangle_{RA'} \Rightarrow \mathcal{E}(\rho) = \sum_a M_a \rho M_a^\dagger$

Isomorphism of states and CP maps.

Summary : $\mathcal{E}_{A\to A'}$ is completely positive $\Rightarrow I\otimes\mathcal{E}$ takes maximally entangled state on RA to density operator on RA'.
Density operator = ensemble of pure states, each associated with the Kraus operator in Kraus representation of $\mathcal{E}$.

Channel-State Duality (Choi-Jamiolkowski Isomorphism)

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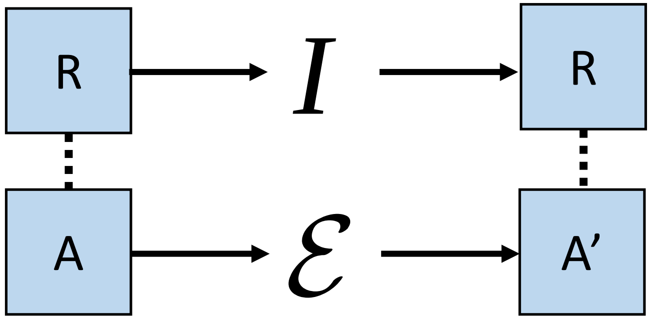

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"Relative state method": $$\mathcal{E}\left(\left(|\varphi\rangle\langle\varphi|\right)_A\right) = \sum_a \left(\langle\varphi^*|\tilde{\Psi}_a\rangle\langle\tilde{\Psi}_a|\varphi^*\rangle\right)_{A'}. = \sum_a M_a|\varphi\rangle\langle\varphi|M_a^\dagger$$

The Kraus representation of a nonunitary channel is not unique, just as the ensemble representation of a mixed-state density operator is not unique. Recalling HJW Theorem:

$$\sum_a \left(|\tilde{\Psi}_a\rangle\langle\tilde{\Psi}_a|\right)_{RA'} = \sum_\mu \left(|\tilde{\gamma}_\mu\rangle\langle\tilde{\gamma}_\mu|\right)_{RA'} \Rightarrow |\tilde{\gamma}_\mu\rangle = \sum_a V_{\mu a}|\tilde{\Psi}_a\rangle, \text{ where } V \text{ is unitary.}$$

Likewise: $$\mathcal{E}(\rho) = \sum_a M_a \rho M_a^\dagger = \sum_\mu N_\mu \rho N_\mu^\dagger \Rightarrow N_\mu = \sum_a V_{\mu a} M_a$$

Channel-State Duality (Choi-Jamiolkowski Isomorphism)

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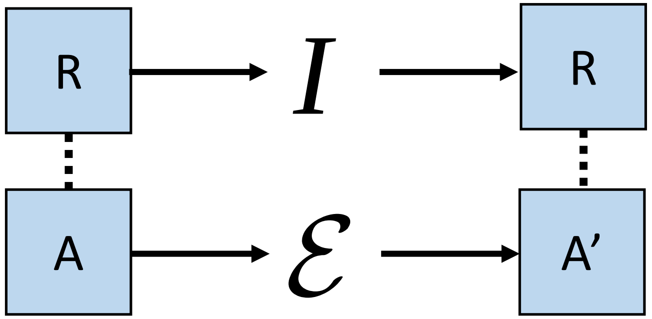

$$\sum_a \left(|\tilde{\Psi}_a\rangle\langle\tilde{\Psi}_a| \right)_{RA'}$$

Normalization depends on *a*.

"Relative state method": $\mathcal{E}\left(\left(|\varphi\rangle\langle\varphi|\right)_A \right) = \sum_a \left(\langle\varphi^* |\tilde{\Psi}_a\rangle\langle\tilde{\Psi}_a|\varphi^*\rangle \right)_{A'} . = \sum_a M_a |\varphi\rangle\langle\varphi| M_a^\dagger$

How many Kraus operator do we need to represent a channel mapping A to A'? As many as the number of pure states needed in ensemble representation of a density operator on RA'.

$d_{RA'} = d_A d_{A'}$ Kraus operators suffice.

How many real parameters needed to specify a quantum channel? It is a linear map of Hermitian operators on A to Hermitian operators on A', which preserves the trace for any input. So # of parameters:

$d_A^2 d_{A'}^2 - d_A^2$

12 parameters for a map of a qubit to a qubit. In contrast, we need only 3 parameters to specify a unitary map.