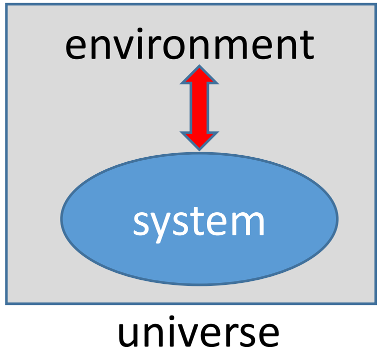
Ph/CS 219A
Quantum Computation

Lecture 5. Qubit Channels



This is the fourth (and last) lecture in which we will develop the theory of *open quantum systems*.

We say that a system is "open" if it can exchange energy and information with its environment.

We have seen that in the setting of open systems, states are density operators, measurements are POVMs, and evolution maps are quantum channels (TPCP maps).

We have discussed the operator sum representation of a channel (Kraus representation) and why physical operations on quantum states should be completely positive. We used an isomorphism between states and completely positive maps to show that a completely positive map always has an operator sum representation.

Our discussion last time of quantum channels was rather general and abstract, so we will flesh it out today with some examples. The examples are instructive, illustrating some important principles.

See Chapter 3 of the Lecture Notes. Reminder: Homework #1 has been posted, due Friday October 16

Qubit depolarizing channel

With probability *1-p*, no error occurs, and with probability *p*, one of three equally likely Pauli errors occurs.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

X is a bit flip ($|0\rangle \leftrightarrow |1\rangle$), Z is a "phase error" ($|+\rangle \leftrightarrow |-\rangle$), and $Y = -iZX$ is both at once.

The Kraus operators for this channel are:

$$M_0 = \sqrt{1-p}\, I, \quad M_1 = \sqrt{\frac{p}{3}}\, X, \quad M_2 = \sqrt{\frac{p}{3}}\, Y, \quad M_3 = \sqrt{\frac{p}{3}}\, Z$$

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}\left(X\rho X + Y\rho Y + Z\rho Z \right)$$

$$X^2 = Y^2 = Z^2 = I \Rightarrow \sum_a M_a^\dagger M_a = (1-p)I + 3(p/3)I = I$$

Qubit depolarizing channel

Channel-state duality:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \quad |\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

$$(\mathcal{E} \otimes I)(|\phi^+\rangle\langle\phi^+|) = (1-p)|\phi^+\rangle\langle\phi^+| + \frac{p}{3}\Big(|\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-| + |\phi^-\rangle\langle\phi^-|\Big)$$

For p = ¾: $$(\mathcal{E} \otimes I)(|\phi^+\rangle\langle\phi^+|) = \frac{1}{4}\Big(|\phi^+\rangle\langle\phi^+| + |\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-| + |\phi^-\rangle\langle\phi^-|\Big) = \frac{1}{4} I \otimes I$$

Unconventional normalization (relative state method):

$$(\mathcal{E} \otimes I)(|\tilde{\phi}^+\rangle\langle\tilde{\phi}^+|) = \frac{1}{2} I \otimes I \Rightarrow \mathcal{E}(|\psi\rangle\langle\psi|) = \langle\psi^*| \frac{1}{2} I \otimes I |\psi^*\rangle = \frac{1}{2} I$$

The depolarizing channel with p=3/4 maps *any* input to the maximally mixed state ("completely depolarizing").

Qubit depolarizing channel

More generally:

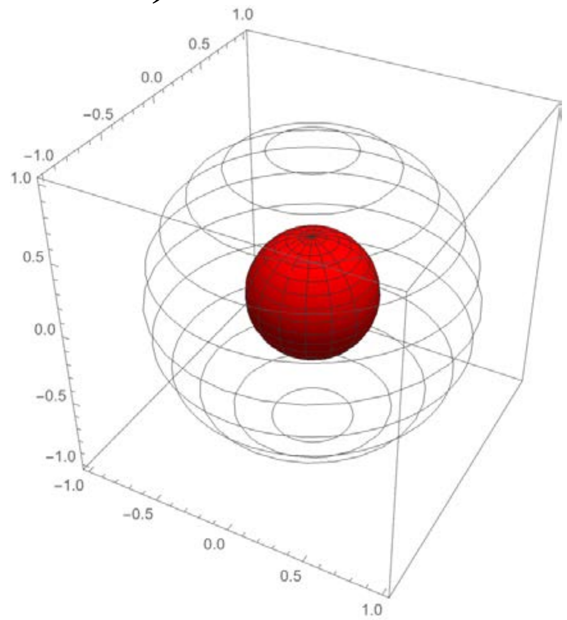
$$(\mathcal{E}\otimes I)(|\phi^+\rangle\langle\phi^+|)=\left(1-\frac{4p}{3}\right)|\phi^+\rangle\langle\phi^+|+\frac{p}{3}\big(|\phi^+\rangle\langle\phi^+|+|\psi^+\rangle\langle\psi^+|+|\psi^-\rangle\langle\psi^-|+|\phi^-\rangle\langle\phi^-|\big)=\left(1-\frac{4p}{3}\right)|\phi^+\rangle\langle\phi^+|+\frac{4p}{3}\left(\frac{1}{4}I\otimes I\right)$$

For $p \leq$ ¾, we discard the input state and replace it by the maximally mixed state with probability $4p/3$.

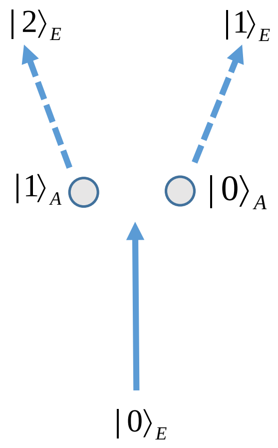
The Bloch sphere deflates:

$$\rho(\vec{P})=\frac{1}{2}\left(I+\vec{P}\cdot\vec{\sigma}\right)\Rightarrow\mathcal{E}(\rho)=\left(1-\frac{4}{3}p\right)\rho+\frac{4}{3}pI=\rho(\vec{P}'),\quad \vec{P}=\left(1-\frac{4}{3}p\right)\vec{P}$$

A map that inverts the deflation is not a channel; it maps polarization contained in the Bloch ball (a physical density operator), to a polarization outside the Bloch ball (not a physical state).



Qubit dephasing channel



Physical picture: A dust grain is in a superposition of two different spatial positions, labeled 0 and 1. A photon is scattered by the dust grain, making a transition from its initial state 0, to one of two possible final states (1 and 2) depending on the position of the dust grain. Hence the dust grain and the photon become entangled.

Isometric map describing the process:

$$|0\rangle_A \mapsto \sqrt{1-p}\;|0\rangle_A \otimes |0\rangle_E + \sqrt{p}\;|0\rangle_A \otimes |1\rangle_E,$$
$$|1\rangle_A \mapsto \sqrt{1-p}\;|1\rangle_A \otimes |0\rangle_E + \sqrt{p}\;|1\rangle_A \otimes |2\rangle_E.$$

Trace out environment:

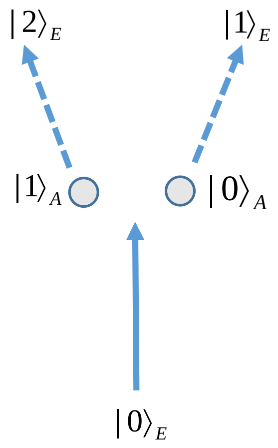
$$M_0 = \sqrt{1-p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \sqrt{p}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

These 3 Kraus operators are not linearly independent --- we can express all three in terms of *I* and *Z*.

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{4}\big((I+Z)\rho(I+Z) + (I-Z)\rho(I-Z)\big) = (1-p)\rho + \frac{p}{2}\big(\rho + Z\rho Z\big) = \left(1-\frac{p}{2}\right)\rho + \frac{p}{2} Z\rho Z$$

A Z error occurs with probability p/2. Dephasing occurs in a *preferred basis*.

Qubit dephasing channel



$$M_0 = \sqrt{1-p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \sqrt{p}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Off-diagonal entries in the density operator decay:

$$\mathcal{E}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix};$$

Continuous dephasing:

Probability of scattering per unit time $\Gamma \Rightarrow p = \Gamma\Delta t \ll 1$. Over time $t = n\Delta t$, $\mathcal{E}^n$ is applied

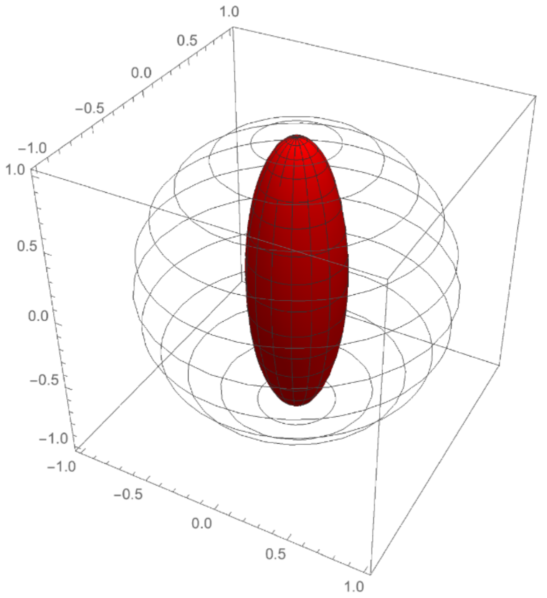
$\Rightarrow$ suppression by $(1-p)^n = (1-\Gamma t/n)^n \to e^{-\Gamma t}$ ($n \to \infty$ with t fixed). Exponential decay.

$$\mathcal{E}^n\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & e^{-\Gamma t}\rho_{01} \\ e^{-\Gamma t}\rho_{10} & \rho_{11} \end{pmatrix}$$

After a time long compared to Γ^{-1}, a coherent superposition $a|0\rangle + b|1\rangle$ decays to mixture

$\rho = |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1|$. Decoherence occurs in a preferred basis.

Qubit dephasing channel

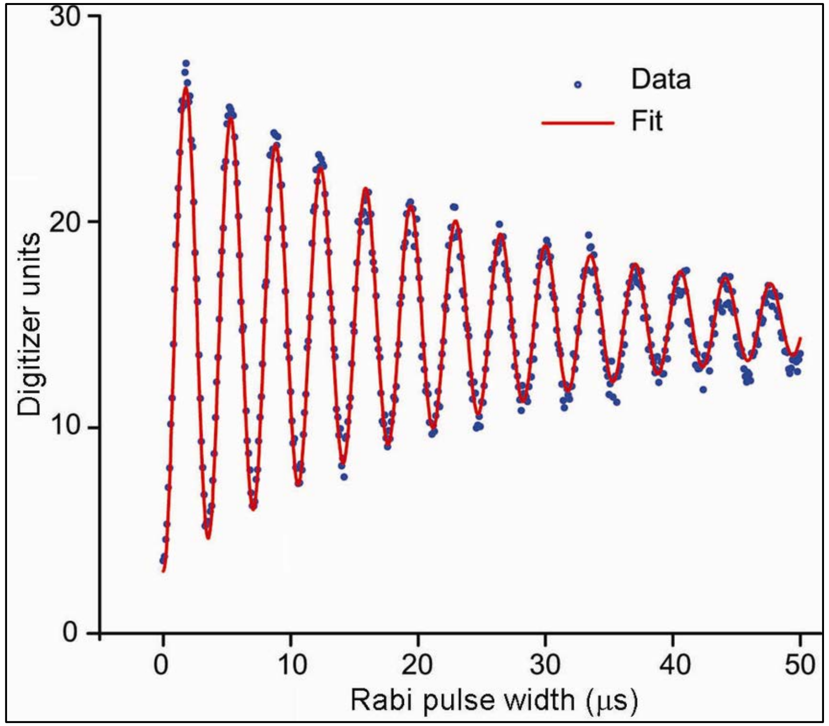


Bloch sphere picture:

$$P_3 \to P_3, \quad P_{1,2} \to \left(1-\frac{p}{2}\right)P_{1,2} - \frac{p}{2}P_{1,2} = (1-p)P_{1,2}$$

But "no pancake". Map that flattens to Bloch sphere to a oblate ellipsoid is not a channel (cf. transpose map).

Decay of *Rabi oscillations*. E.g. an electron spin, trapped atom, …



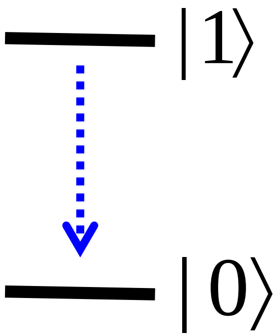
$$H/\hbar = \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix} \Rightarrow |\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\omega t}|1\rangle\right) \quad \text{(if no dephasing)}$$

$$\rho(t) = \frac{1}{2}\begin{pmatrix} 1 & e^{i\omega t}e^{-\Gamma t} \\ e^{-i\omega t}e^{-\Gamma t} & 1 \end{pmatrix} \Rightarrow \mathrm{Prob}(+,t) = \langle + |\rho(t)| + \rangle = \frac{1}{2}\left(1 + e^{-\Gamma t}\cos \omega t\right)$$

Visibility of interference decays exponentially in time. We can measure the dephasing rate by fitting the exponential.

Qubit amplitude-damping channel (spontaneous decay)

With probability *p*, an excited atom decays to its ground state by emitting a photon. The "environment" is the electromagnetic field; its state changes from "no photon" to "one photon" when emission occurs.



$$|0\rangle_A \otimes |0\rangle_E \mapsto \quad |0\rangle_A \otimes |0\rangle_E$$

$$|1\rangle_A \otimes |0\rangle_E \mapsto \sqrt{1-p}\ |1\rangle_A \otimes |0\rangle_E + \sqrt{p}\ |0\rangle_A \otimes |1\rangle_E .$$

Two Kraus operators: "jump" and "no jump."

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \mathcal{E}(\rho) = M_0 \rho M_0^\dagger + M_1 \rho M_1^\dagger = \begin{pmatrix} \rho_{00} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & (1-p)\rho_{11} \end{pmatrix} + \begin{pmatrix} p\rho_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & (1-p)\rho_{11} \end{pmatrix}$$

Spontaneous decay affects both the on-diagonal and the off-diagonal entries.

Qubit amplitude-damping channel (spontaneous decay)

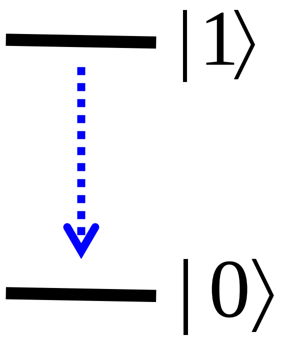
$$\mathcal{E}(\rho) = M_0 \rho M_0^\dagger + M_1 \rho M_1^\dagger = \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & (1-p)\rho_{11} \end{pmatrix}$$

$$p = \Gamma \Delta t \ll 1 \Rightarrow (1-p)^n = (1-\Gamma t/n)^n \to e^{-\Gamma t}$$

$$\rho(t) = \begin{pmatrix} \rho_{00} + \left(1 - e^{-\Gamma t}\right)\rho_{11} & e^{-\Gamma t/2}\rho_{01} \\ e^{-\Gamma t/2}\rho_{10} & e^{-\Gamma t}\rho_{11} \end{pmatrix}$$

Decay of excited population: $e^{-t/T_1}, \quad T_1 = \Gamma^{-1}$

Decay of off-diagonal term: $e^{-t/T_2}, \quad T_2 = 2\Gamma^{-1} = 2T_1$



Observing the environment

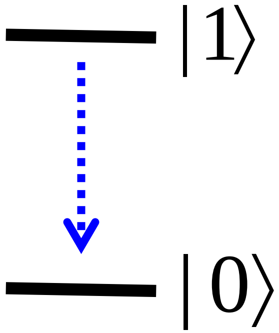
Surround the atom with photodetectors; when an emitted photon is detected, we know the atom has decayed to its ground state. If no photon is detected as time goes by, we know *a posteriori* that the atom was most likely in its ground state to begin with – otherwise we should have seen it decay by now!

$$(\alpha|0\rangle_A + \beta|1\rangle_A)\otimes|0\rangle_E \mapsto (\alpha|0\rangle_A + \beta\sqrt{1-p}\;|1\rangle_A)\otimes|0\rangle_E + \beta\sqrt{p}\;|0\rangle_A\otimes|1\rangle_E$$

Consider n consecutive time intervals. M_0 is the Kraus operator for no detection, M_k is the Kraus operator for detection in the kth interval.

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{(1-p)^n} \end{pmatrix}, \quad M_k = \begin{pmatrix} 0 & \sqrt{(1-p)^{k-1}p} \\ 0 & 0 \end{pmatrix}$$

$$M_0\left(\alpha|0\rangle + \beta|1\rangle\right) = \alpha|0\rangle + e^{-\Gamma t/2}\beta|1\rangle$$



A (destructive) orthogonal measurement if we wait long enough: if a photon is detected we know the atom was in its excited state, if no photon is detected we know the atom was in its ground state.

Master Equation

For a closed system, continuous time evolution is unitary, governed by Schroedinger equation.

$$|\psi(t+dt) = \left(I - iHdt\right)|\psi(t)\rangle$$

For an open system, continuous *Markovian* time evolution is not unitary, governed by a *master equation*.

$$\rho(t+dt) = \mathcal{E}_{dt}(\rho(t)) = \sum_a M_a \rho(t) M_a^\dagger = \rho(t) + O(dt)$$

Why Markovian? Implicit assumption is that the environment "forgets quickly" and is continuously refreshed. So the change in the density operator in each time step is a channel (TPCP map).

Imagine the environment is measured after an infinitesimal time step. Either there is "no jump" (in which case an O(dt) modification of the system occurs), or there is a "jump" occurring with probability O(dt).

$$M_0 = I + (-iH + K)dt, \quad M_a = \sqrt{dt}\, L_a \quad (a > 0),$$

Master Equation

$$\rho(t+dt) = \mathcal{E}_{dt}(\rho(t)) = \sum_a M_a \rho(t) M_a^\dagger = \rho(t) + O(dt)$$

$$M_0 = I + (-iH + K)dt, \quad M_a = \sqrt{dt}\, L_a \quad (a > 0),$$

This map is completely positive (has an operator sum normalization). We should also enforce that it is trace preserving.

$$I = \sum_a M_a^\dagger M_a = I + dt\left(2K + \sum_{a>0} L_a^\dagger L_a\right) + \cdots \quad \Rightarrow \quad K = -\frac{1}{2}\sum_{a>0} L_a^\dagger L_a$$

$$\boxed{\dot{\rho} = \mathcal{L}(\rho) = -i[H,\rho] + \sum_{a>0}\left(L_a \rho L_a^\dagger - \frac{1}{2} L_a^\dagger L_a \rho - \frac{1}{2}\rho L_a^\dagger L_a\right).}$$

Lindblad master equation, preserves the trace. As with any nonunitary channel, we have the freedom to choose the jump operators in various ways (*unravelings* of the master equation) --- different possible measurements of environment we might have performed.