Ph/CS 219A
Quantum Computation

Lecture 7. General Bell Inequalities

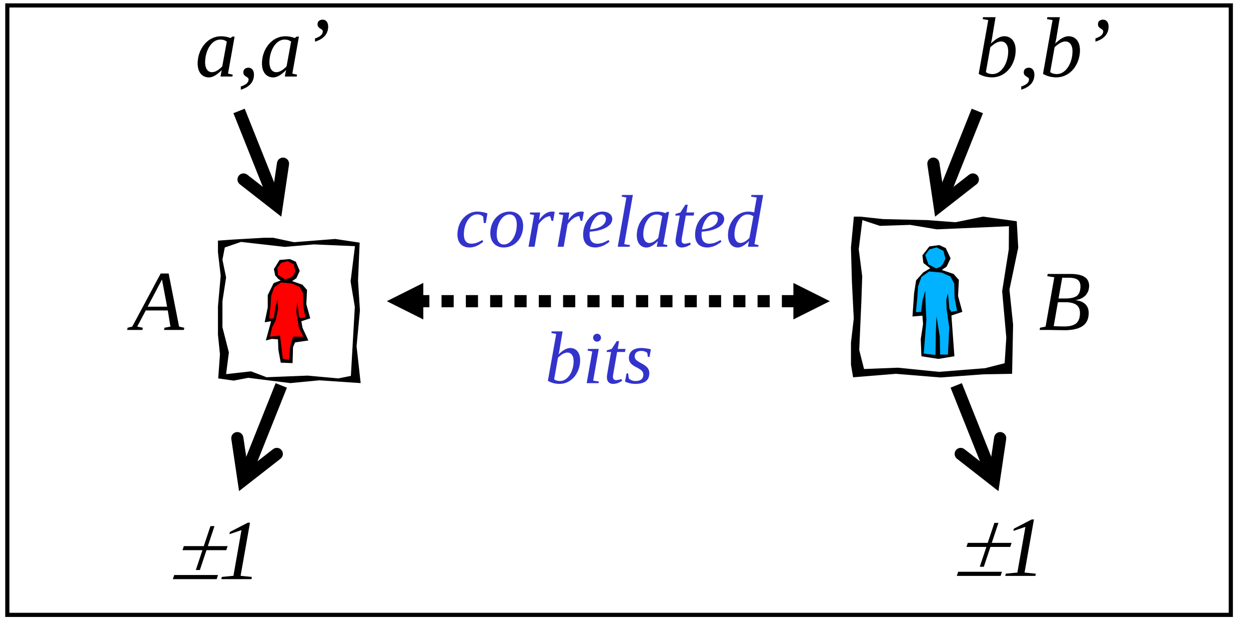
Last time we discussed the CHSH inequality, which applies to classical strategies in a two-party cooperative game, and we say that quantum strategies (exploiting shared quantum entanglement) can violate this inequality. Violation of the CHSH inequality has been experimentally confirmed. Quantum correlations, stronger than classical ones, are part of Nature's design.

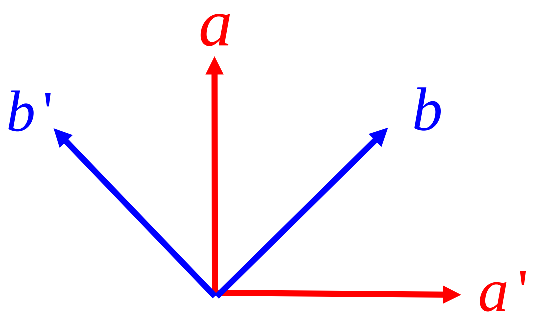
Today we continue the discussion of quantum entanglement and Bell inequalities. We will put Bell inequalities in a more general context, and then revisit the CHSH inequality from a fresh viewpoint.

This lecture will depart from the typed lecture notes, but I hope the slides are self-contained. For further reading, see:
R. F. Werner and M. M Wolf, All multipartite Bell correlation inequalities for two dichotomic observables per site,
https://arxiv.org/abs/quant-ph/0102024

See Chapter 4 of the Lecture Notes. Note that Problem Set 2 has been posted, due October 30.

Clauser-Horne-Shimony-Holt (CHSH) Inequality





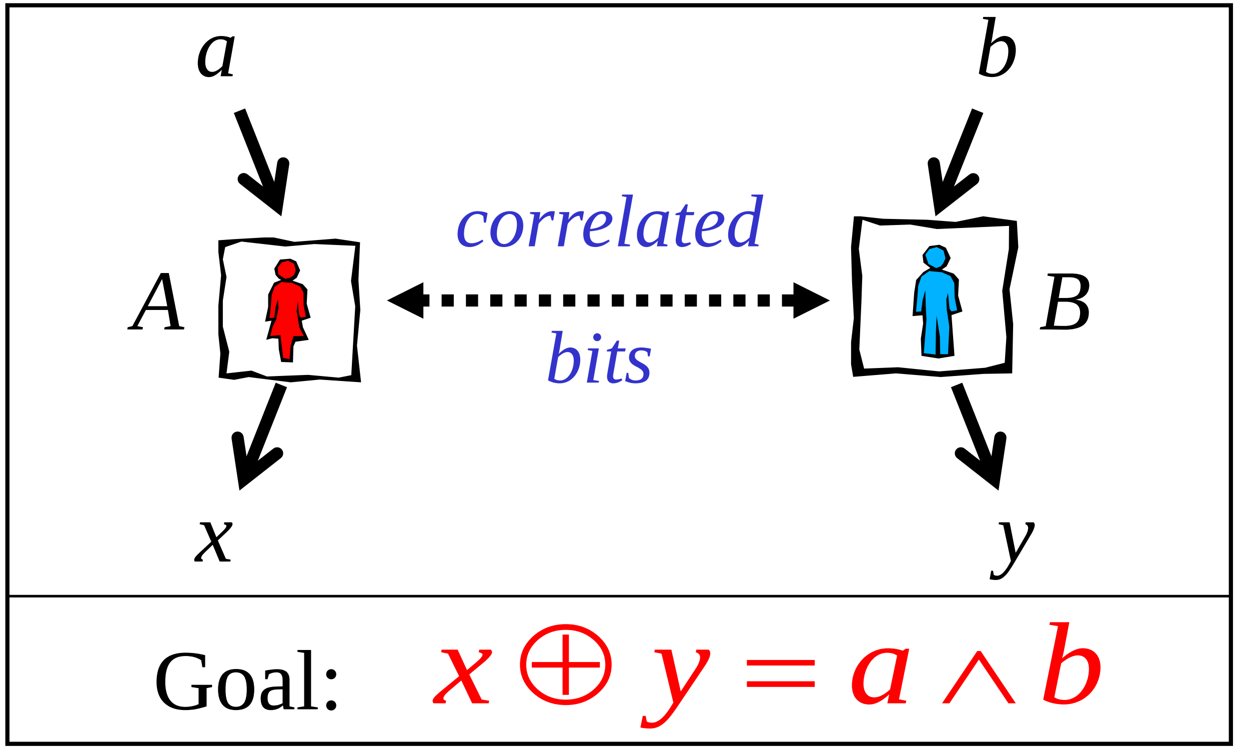
$$|\langle C\rangle| = |\langle ab\rangle + \langle a'b\rangle + \langle ab'\rangle - \langle a'b'\rangle| \le 2$$ CHSH inequality (classical)

$$\langle\psi^-|\vec{\sigma}\cdot\hat{a}\otimes\vec{\sigma}\cdot\hat{b}|\psi^-\rangle = -\hat{a}\cdot\hat{b} = -\cos\theta$$

$$\Rightarrow |\langle C\rangle| = \left|-3\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right| = 2\sqrt{2} > 2.$$ The CHSH inequality is (maximally) violated.

$$|\langle C\rangle| \le 2\sqrt{2}$$ Cirel'son inequality (quantum)

CHSH Game



$p(a,b)$ is probability of winning for inputs a,b.

$$\Rightarrow \frac{1}{4}\big(p(0,0)+p(0,1)+p(1,0)+p(1,1)\big) \le 3/4$$

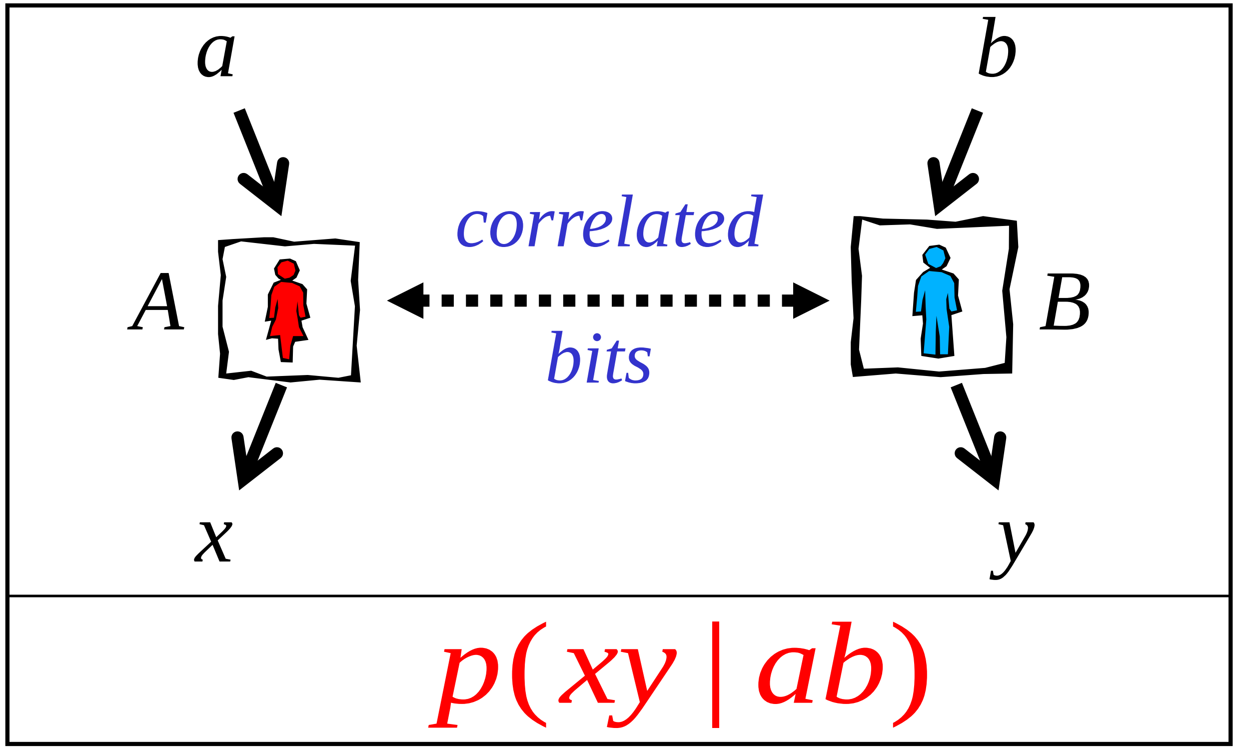
(classical)

$$\frac{1}{4}\big(p(0,0)+p(0,1)+p(1,0)+p(1,1)\big) \le \frac{1}{2}+\frac{1}{2\sqrt{2}} = .853$$

(quantum)

Averaged uniformly over inputs, no “classical strategy” can win the game with success probability better than .75, while for “quantum strategies”, the highest possible success probability improves to .853.

Bell inequalities more generally



Now *a, b* are possible measurement "settings" for Alice and Bob, taking *m* possible values.

And *x, y* are possible measurement "outcomes" taking *v* possible values.

$p(xy\,|\,ab)$ denotes the conditional probability of outcomes x, y when settings are a, b. We call $p(xy\,|\,ab)$ a "state" and denote the space of possible states by Ω.

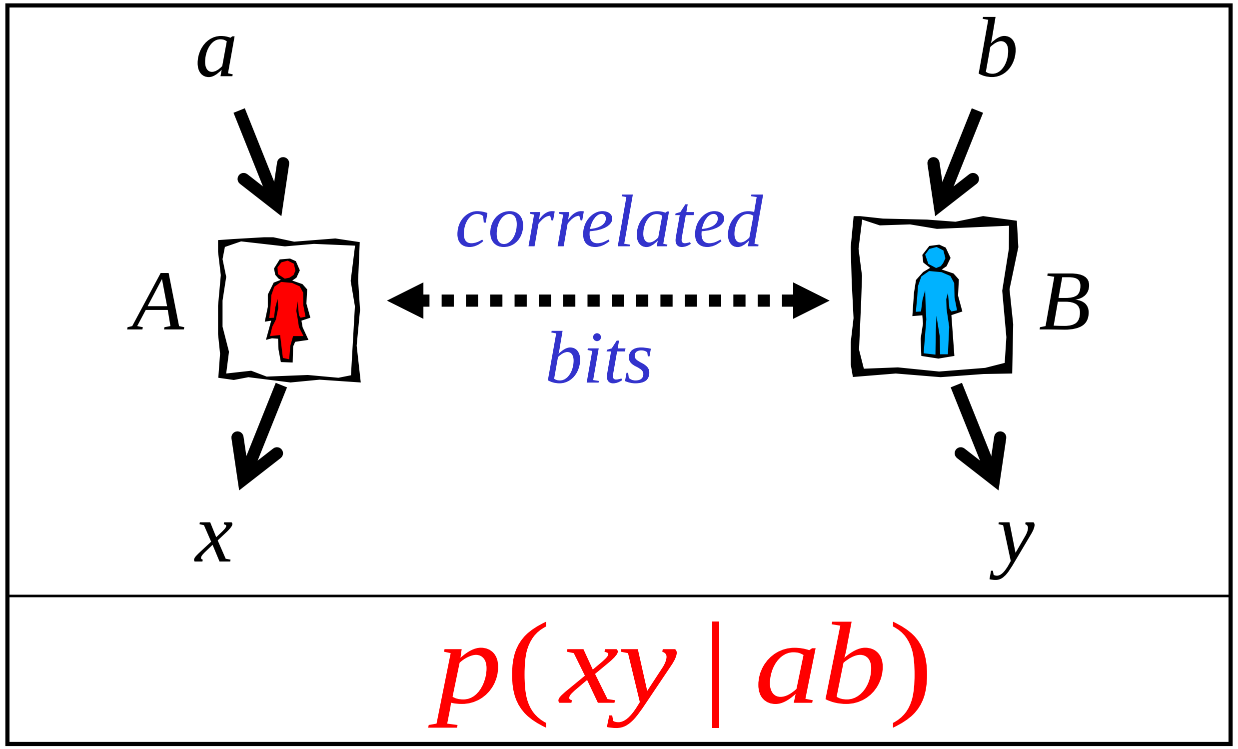
For any choice of settings, the probabilities of outcomes sum to 1: $\sum_{x,y} p(xy\,|\,ab) = 1$ for all a, b.

Therefore, the state space has dimension $|\Omega| = m^2(v^2 - 1)$. For example, $|\Omega| = 12$ for $m = v = 2$.

In a "local configuration" there is a definite *x* for each *a* (which does not depend on *b*), and a definite *y* for each *b* (which does not depend on *a*).

A "local model" also known as a "local hidden variable theory" (LHVT) assigns a probability to each local configuration. *Bell inequalities* are constraints on *p(xy|ab)* that apply to any local model.

Bell inequalities more generally



$p(xy\,|\,ab)$ denotes the conditional probability of outcomes x, y when settings are a, b.

$$\sum_{x,y} p(xy\,|\,ab) = 1 \quad \text{for all } a, b.$$

State space Ω has dimension $|\Omega| = m^2(v^2 - 1)$.

In a "local configuration" there is a definite *x* for each *a* (which does not depend on *b*), and a definite *y* for each *b* (which does not depend on *a*).

Local configuration: $p(xy\,|\,ab) = \delta(x, x_a)\delta(y, y_b)$. There are v choices for each x_a, y_b; hence v^{2m} local configurations.

Contrast with *nonlocal* configuration. Now outcomes can depend on both settings: $p(xy\,|\,ab) = \delta(x, x_{ab})\delta(y, y_{ab})$.

There are v choices for each x_{ab}, y_{ab}; hence v^{2m^2} local configurations.

Example: $m = v = 2 \Rightarrow$ state space dimension $|\Omega| = 12$, 16 local configurations, 256 nonlocal configurations.

A "local model" also known as a "local hidden variable theory" (LHVT) assigns a probability to each local configuration. *Bell inequalities* are constraints on *p(xy|ab)* that apply to any local model.

Geometry of local models (Bell polytope)

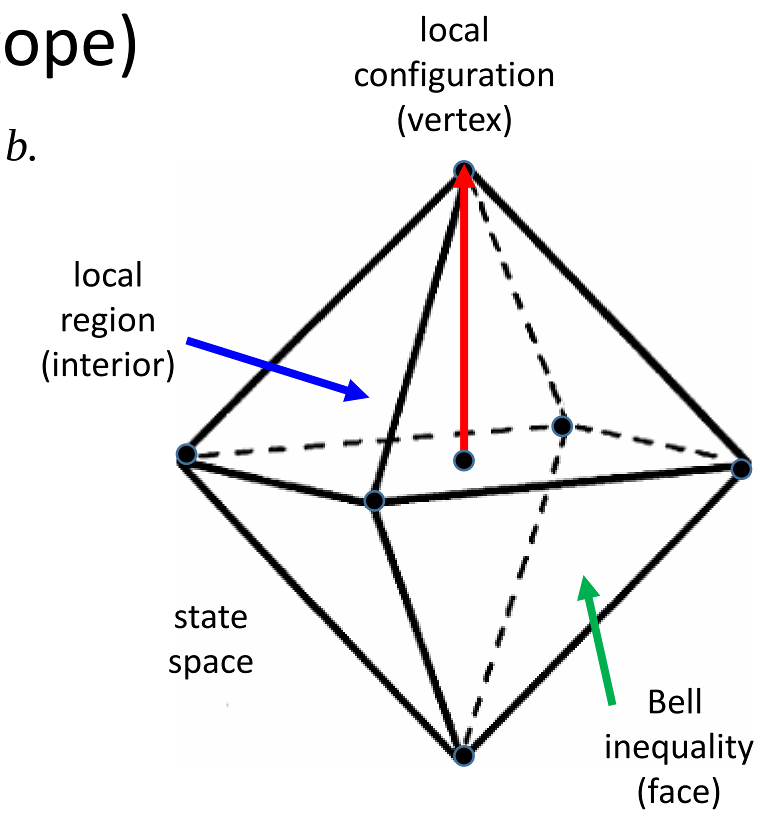
$p(xy\,|\,ab)$ denotes the conditional probability of outcomes x, y when settings are a, b.

A *local model* assigns a probability to each local configuration.

The *local region* (also known as the *classical region*) is the *convex hull* of local configurations $\mathcal{C} \subset \Omega$: the *Bell polytope*.

The *origin* in state space is the uniformly weighted sum of local configs.

Associate with each local config a vector pointing from origin to that point.



Suppose we do experiments to determine the state *p(xy|ab)* to high statistical accuracy. We would like to know whether this state is in the classical region --- can the results be explained by a classical local model?

Give a point $\xi \in \Omega$, is $\xi \in \mathcal{C}$? That might not be easy to answer.

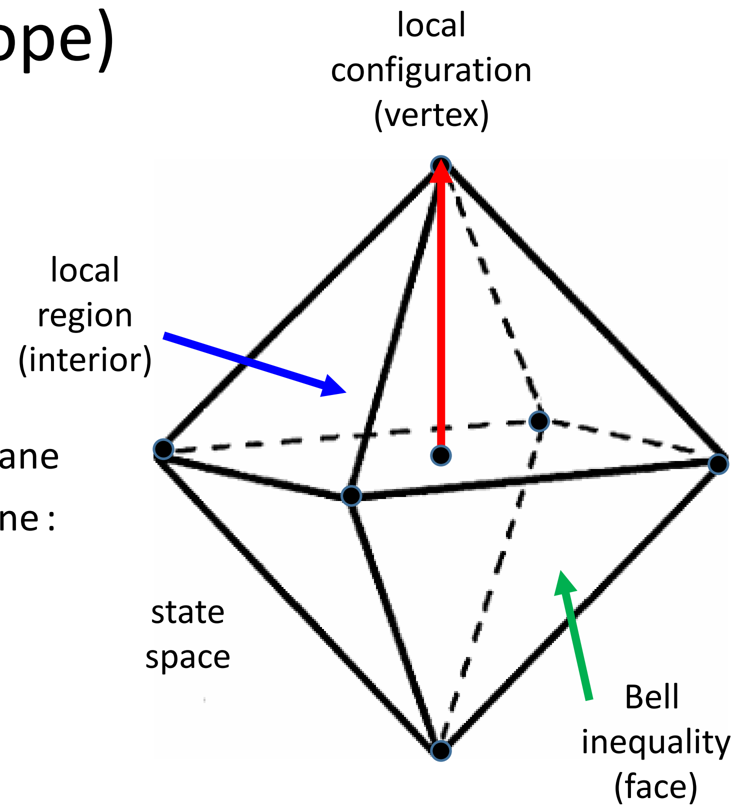
The experimentalist might find it useful to know an inequality, such that if the inequality is violated then we know for sure that $\xi \notin \mathcal{C}$ (a *Bell inequality*).

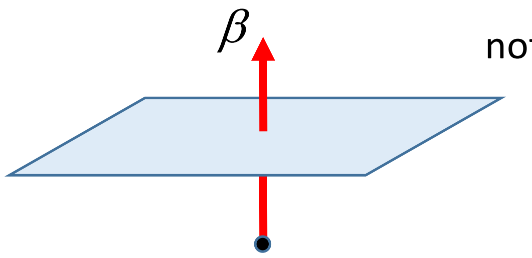
She can easily check whether her data satisfies the inequality or not.

A Bell inequality is a *hyperplane* in the state space, such that the classical region lies on one side of the hyper plane. A *tight* Bell inequality is a plane that contains a face of the Bell polytope. There is a dual description of the polytope in terms of extremal inequalities instead of local configurations; that is, in terms of faces rather than vertices in the state space.

Geometry of local models (Bell polytope)

A Bell inequality is a *plane* in the state space, such that the classical region lies on one side of the plane. A *tight* Bell inequality is a plane that contains a face of the Bell polytope. There is a dual description of the polytope in terms of extremal inequalities instead of local configurations; that is, in terms of faces rather than vertices in the state space.





In a d-dimensional inner-product space, a $(d-1)$-dimensional hyperplane not through the origin is specified by a vector normal to the hyperplane:

$$\{\xi \in \Omega \mid \xi \cdot \beta = 1\}.$$

The half space containing the origin bounded by the hyperplane is

$$\{\xi \in \Omega \mid \xi \cdot \beta \leq 1\}.$$

The *polar* $\mathcal{B}$ of the polytope $\mathcal{C}$ is the set of half spaces that contain $\mathcal{C}$:

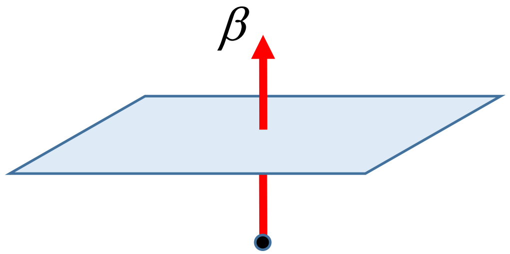
$$\mathcal{B} = \{\beta \in \Omega \mid \beta \cdot \xi \leq 1 \quad \forall \xi \in \mathcal{C}\}$$

$$\mathcal{C} = \{\xi \in \Omega \mid \beta \cdot \xi \leq 1 \quad \forall \beta \in \mathcal{B}\}.$$

We say that $\mathcal{B}$ and $\mathcal{C}$ are *dual* polytopes.

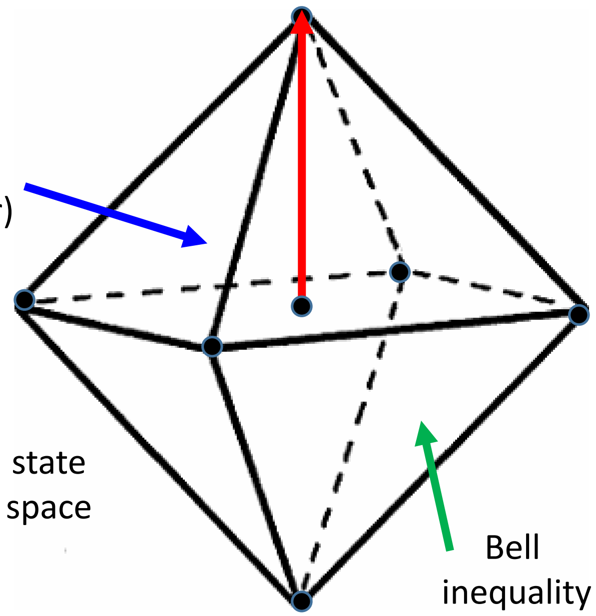
You can check that the polar is convex. Furthermore, the extremal points of the primal polytope correspond to the faces of its dual polytope. Each face of the classical region (each point of its polar) is a tight Bell inequality, asserting that a point in the classical region lies on the same side of that face as the origin. The complete set of Bell inequalities (all faces of the Bell polytope) completely characterize the classical region.

Geometry of local models (Bell polytope)



The complete set of Bell inequalities (extremal points of the dual polytope) fully characterize the classical region.

$$\mathcal{C} = \{\xi \in \Omega \,|\, \xi \cdot \beta^{(i)} \le 1 \quad \forall \text{ extremal } \beta^{(i)} \in \mathcal{B}\}$$



Give a point in state space, it is easy to check whether the state satisfies a given Bell inequality. Unfortunately, though, finding the polar of a polytope with many vertices is computationally difficult (the best known algorithm has a runtime which is superpolynomial in the number of extremal points of the primal polytope).

The problem of computing the polar of the Bell polytope *has* been solved in a few simple cases, such as *m=v=2* (16 vertices in a 12-dimensional state space). It helps that the Bell polytope has a lot of symmetry, but even for the simplest cases, the polytope is hard to visualize!

Life is easier if we reduce the dimensionality by considering *correlators*, expectation values of observables that capture information about how Alice's and Bob's systems are correlated. In the case *m=v=2*, for example, we may consider the four-dimensional space parametrized by

$\left(\langle ab\rangle, \langle ab'\rangle, \langle a'b\rangle, \langle a'b'\rangle\right)$. In a local configuration, each component takes a value ± 1;

there are 8 extremal points, because simultaneously flipping signs of a, a', b, b' does not change the correlator.

CHSH Revisited

What is the classical region for the CHSH correlator $\left(\langle ab\rangle,\langle ab'\rangle,\langle a'b\rangle,\langle a'b'\rangle\right)$?

In a local configuration, each of the 4 components takes a value ± 1; there are 8 extremal points, because simultaneously flipping signs of a,a',b,b' does not change the correlator.

In fact, if a,a',b,b' take values in $\{\pm 1\}$, then $(ab)(ab')(a'b)(a'b')=1$. Extremal points $\{\xi\}$ have even parity:

$$(+,+,+,+) \quad (-,-,-,-)$$
$$(+,+,-,-) \quad (-,-,+,+)$$
$$(+,-,+,-) \quad (-,+,-,+)$$
$$(+,-,-,+) \quad (-,+,+,-)$$

Therefore, if β is in the polar, it satisfies 8 inequalities:

$$-1 \le \beta_0+\beta_1+\beta_2+\beta_3 \le 1$$
$$-1 \le \beta_0+\beta_1-\beta_2-\beta_3 \le 1$$
$$-1 \le \beta_0-\beta_1+\beta_2-\beta_3 \le 1$$
$$-1 \le \beta_0-\beta_1-\beta_2+\beta_3 \le 1$$

Extremal values of β occur when inequalities are saturated; each linear combination equals either $+1$ or -1.

For *extremal* points of the polar:

$$\beta_0+\beta_1+\beta_2+\beta_3 = f_0$$
$$\beta_0+\beta_1-\beta_2-\beta_3 = f_1$$
$$\beta_0-\beta_1+\beta_2-\beta_3 = f_2$$
$$\beta_0-\beta_1-\beta_2+\beta_3 = f_3$$

where $f_0, f_1, f_2, f_3 \in \{\pm 1\}$

There are 16 possible choices for $\{f_i\}$. For each choice, we solve for β, finding the corresponding extremal point of the polar.

$$f=(+,+,+,+) \Rightarrow \beta=(1,0,0,0) \qquad f=(-,-,-,-) \Rightarrow \beta=(-1,0,0,0)$$
$$f=(+,+,-,-) \Rightarrow \beta=(0,1,0,0) \qquad f=(-,-,+.+) \Rightarrow \beta=(0,-1,0,0)$$
$$f=(+,-,+,-) \Rightarrow \beta=(0,0,1,0) \qquad f=(-,+,-,+) \Rightarrow \beta=(0,0,-1,0)$$
$$f=(+,-,-,+) \Rightarrow \beta=(0,0,0,1) \qquad f=(-,+,+,-) \Rightarrow \beta=(0,0,0,-1)$$

$$\boxed{\Rightarrow -1 \le \langle ab\rangle \le 1}$$

(and same for other 3 components of correlator). Here f has even parity; these Bell inequalities are not very informative!

CHSH Revisited

What is the classical region for the CHSH correlator $\left(\langle ab\rangle, \langle ab'\rangle, \langle a'b\rangle, \langle a'b'\rangle\right)$?

For *extremal* points of the polar:

$$\beta_0+\beta_1+\beta_2+\beta_3 = f_0$$
$$\beta_0+\beta_1-\beta_2-\beta_3 = f_1$$
$$\beta_0-\beta_1+\beta_2-\beta_3 = f_2$$
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where $f_0, f_1, f_2, f_3 \in \{\pm 1\}$

$$f=(+,+,+,+) \Rightarrow \beta=(1,0,0,0) \qquad f=(-,-,-,-) \Rightarrow \beta=(-1,0,0,0)$$
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$$f=(+,-,-,+) \Rightarrow \beta=(0,0,0,1) \qquad f=(-,+,+,-) \Rightarrow \beta=(0,0,0,-1)$$

Here f has even parity; these Bell inequalities are not very informative! (Each component of correlator ranges from -1 to +1.)

$$\Rightarrow -1 \le \langle ab\rangle \le 1$$

(and same for the other three components of the correlator)

$$f=(+,+,+,-) \Rightarrow \beta=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) \qquad f=(-,-,-,+) \Rightarrow \beta=\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$$
$$f=(+,+,-,+) \Rightarrow \beta=\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right) \qquad f=(-,-,+,-) \Rightarrow \beta=\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$$
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$$f=(-,+,+,+) \Rightarrow \beta=\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right) \qquad f=(+,-,-,-) \Rightarrow \beta=\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$$

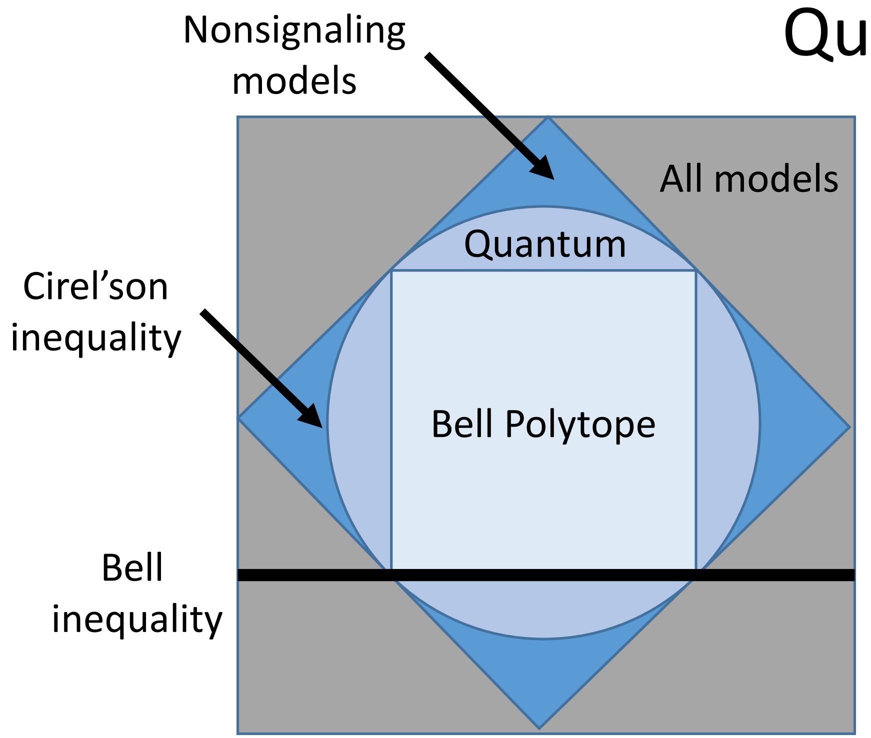
$$\Rightarrow -1 \le \frac{1}{2}\left(\langle ab\rangle + \langle ab'\rangle + \langle a'b\rangle - \langle a'b'\rangle\right) \le 1$$

(and three other possible places for the minus sign)

These 16 inequalities completely characterize the Bell polytope. If any one of the 8 CHSH-like inequalities is violated (they are all related to our earlier CHSH inequality by symmetry), then the correlator cannot be explained by any classical local model.

Here f has odd parity. These are the nontrivial Bell inequalities.

Quantum and classical regions



There is also a “quantum region” --- Alice and Bob share a density operator, and each performs one of *m* POVMs, each with *v* possible outcomes.

Alice POVM $M_A^{(a)} = \{E_x^{(a)}\}, \quad a = 0,1,\ldots,m-1, \quad x = 0,1,\ldots v-1,$

Bob POVM $M_B^{(b)} = \{E_y^{(b)}\}, \quad b = 0,1,\ldots,m-1, \quad y = 0,1,\ldots v-1.$

$$\mathcal{Q} = \{ p(xy\,|\,ab) = \mathrm{tr}\left(E_x^{(a)} \otimes E_y^{(b)}\right)\rho_{AB} \}$$

$\mathcal{Q}$ contains $\mathcal{C}$ but is larger than $\mathcal{C}$.

$\mathcal{Q}$ is contained in the set $\mathcal{N}$ of *nonsignaling* models.

“Nonsignaling” means that Alice’s outcome reveals no information about Bob’s setting and vice versa.

$\sum_y p(xy\,|\,ab) = P_A(x\,|\,a)$ (does not depend on b),

$\sum_x p(xy\,|\,ab) = P_B(y\,|\,b)$ (does not depend on a).

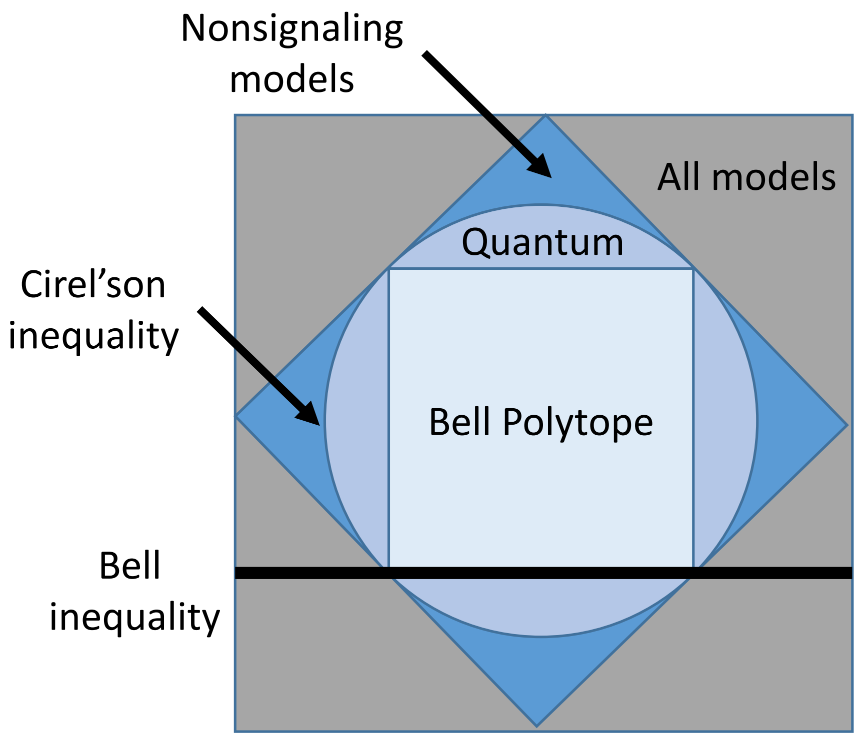
This too is a proper inclusion. There are nonsignaling models in which one can win the CHSH game with probability 1 for any input, violating Cirel’son bound.

Input $(a,b) = (0,0),(0,1),(1,0) \Rightarrow$ Output $(x,y) = (0,0),(1,1)$ equiprobably,

Input $(a,b) = (1,1) \Rightarrow$ Output $(x,y) = (0,1),(1,0)$ equiprobably.

“Nonlocal box”: For any input, Alice outputs 0 and 1 equiprobably, and same for Bob. Quantum correlations are weaker than “causality” would allow.

Quantum and classical regions



Facts about quantum models:

The quantum region contains the classical region.

The quantum region is contained in the nonsignaling region.

The quantum region is convex. But it is *not* a polytope. (The nonsignaling region is also convex, and *is* a polytope.)

If the density operator shared by Alice and Bob is separable (a convex combination of product states), then the model lies in the classical region.

If the parties choose among measurements that are mutually commuting, then the model lies in the classical region.

For Bell inequality violation, we need both entanglement and noncommuting measurements.

We have focused our discussion on the case of two parties, but similar considerations apply to any number of (at least two) parties.