Chapter 3

Foundations II: Measurement and Evolution

3.1 Orthogonal Measurement and Beyond

3.1.1 Orthogonal Measurements

We would like to examine the properties of the *generalized* measurements that can be realized on system A by performing orthogonal measurements on a larger system that contains A. But first we will briefly consider how (orthogonal) measurements of an arbitrary observable can be achieved in principle, following the classic treatment of Von Neumann.

To measure an observable $\mathbf{M}$, we will modify the Hamiltonian of the world by turning on a coupling between that observable and a "pointer" variable that will serve as the apparatus. The coupling establishes entanglement between the eigenstates of the observable and the distinguishable states of the pointer, so that we can prepare an eigenstate of the observable by "observing" the pointer.

Of course, this is not a fully satisfying model of measurement because we have not explained how it is possible to measure the pointer. Von Neumann's attitude was that one can see that it is possible in principle to correlate the state of a microscopic quantum system with the value of a macroscopic classical variable, and we may take it for granted that we can perceive the value of the classical variable. A more complete explanation is desirable and possible; we will return to this issue later.

We may think of the pointer as a particle that propagates freely apart

from its tunable coupling to the quantum system being measured. Since we intend to measure the position of the pointer, it should be prepared initially in a wavepacket state that is narrow in position space — but not too narrow, because a vary narrow wave packet will spread too rapidly. If the initial width of the wave packet is Δx, then the uncertainty in it velocity will be of order $\Delta v = \Delta p/m \sim \hbar/m\Delta x$, so that after a time t, the wavepacket will spread to a width

$$\Delta x(t) \sim \Delta x + \frac{\hbar t}{m\Delta x}, \tag{3.1}$$

which is minimized for $[\Delta x(t)]^2 \sim [\Delta x]^2 \sim \hbar t/m$. Therefore, if the experiment takes a time t, the resolution we can achieve for the final position of the pointer is limited by

$$\Delta x \gtrsim (\Delta x)_{SQL} \sim \sqrt{\frac{\hbar t}{m}}, \tag{3.2}$$

the "standard quantum limit." We will choose our pointer to be sufficiently heavy that this limitation is not serious.

The Hamiltonian describing the coupling of the quantum system to the pointer has the form

$$\mathbf{H} = \mathbf{H}_0 + \frac{1}{2m}\mathbf{P}^2 + \lambda\mathbf{M}\mathbf{P}, \tag{3.3}$$

where $\mathbf{P}^2/2m$ is the Hamiltonian of the free pointer particle (which we will henceforth ignore on the grounds that the pointer is so heavy that spreading of its wavepacket may be neglected), H_0 is the unperturbed Hamiltonian of the system to be measured, and λ is a coupling constant that we are able to turn on and off as desired. The observable to be measured, $\mathbf{M}$, is coupled to the momentum $\mathbf{P}$ of the pointer.

If $\mathbf{M}$ does not commute with $\mathbf{H}_0$, then we have to worry about how the observable evolves during the course of the measurement. To simplify the analysis, let us suppose that either $[\mathbf{M}, \mathbf{H}_0] = 0$, or else the measurement is carried out quickly enough that the free evolution of the system can be neglected during the measurement procedure. Then the Hamiltonian can be approximated as $\mathcal{H} \simeq \lambda\mathbf{M}\mathbf{P}$ (where of course $[\mathbf{M}, \mathbf{P}] = 0$ because $\mathbf{M}$ is an observable of the system and $\mathbf{P}$ is an observable of the pointer), and the time evolution operator is

$$\mathbf{U}(t) \simeq \exp[-i\lambda t\mathbf{M}\mathbf{P}]. \tag{3.4}$$

Expanding in the basis in which $\mathbf{M}$ is diagonal,

$$\mathbf{M} = \sum_a |a\rangle M_a \langle a|, \tag{3.5}$$

we express $\mathbf{U}(t)$ as

$$\mathbf{U}(t) = \sum_a |a\rangle \exp[-i\lambda t M_a \mathbf{P}]\langle a|. \tag{3.6}$$

Now we recall that $\mathbf{P}$ generates a translation of the *position* of the pointer: $\mathbf{P} = -i\frac{d}{dx}$ in the position representation, so that $e^{-ix_o\mathbf{P}} = \exp\left(-x_o\frac{d}{dx}\right)$, and by Taylor expanding,

$$e^{-ix_o\mathbf{P}}\psi(x) = \psi(x - x_o); \tag{3.7}$$

In other words $e^{-ix_o\mathbf{P}}$ acting on a wavepacket translates the wavepacket by x_o. We see that if our quantum system starts in a superposition of $\mathbf{M}$ eigenstates, initially unentangled with the position-space wavepacket $|\psi(x)$ of the pointer, then after time t the quantum state has evolved to

$$\mathbf{U}(t)\left(\sum_a \alpha_a|a\rangle \otimes |\psi(x)\rangle\right)$$

$$= \sum_a \alpha_a|a\rangle \otimes |\psi(x - \lambda t M_a)\rangle; \tag{3.8}$$

the position of the pointer is now correlated with the value of the observable $\mathbf{M}$. If the pointer wavepacket is narrow enough for us to resolve all values of the M_a that occur ($\Delta x \lesssim \lambda t \Delta M_a$), then when we observe the position of the pointer (never mind how!) we will prepare an eigenstate of the observable. With probability $|\alpha_a|^2$, we will detect that the pointer has shifted its position by $\lambda t M_a$, in which case we will have prepared the $\mathbf{M}$ eigenstate $|a\rangle$. In the end, then, we conclude that the initial state $|\varphi\rangle$ or the quantum system is projected to $|a\rangle$ with probability $|\langle a|\varphi\rangle|^2$. This is Von Neumann's model of orthogonal measurement.

The classic example is the Stern–Gerlach apparatus. To measure $\boldsymbol{\sigma}_3$ for a spin-$\frac{1}{2}$ object, we allow the object to pass through a region of inhomogeneous magnetic field

$$B_3 = \lambda z. \tag{3.9}$$

The magnetic moment of the object is $\mu\vec{\boldsymbol{\sigma}}$, and the coupling induced by the magnetic field is

$$H = -\lambda\mu\mathbf{z}\boldsymbol{\sigma}_3. \tag{3.10}$$

In this case $\boldsymbol{\sigma}_3$ is the observable to be measured, coupled to the position $\mathbf{z}$ rather than the momentum of the pointer, but that's all right because $\mathbf{z}$ generates a translation of $\mathbf{P}_z$, and so the coupling imparts an *impulse* to the pointer. We can perceive whether the object is pushed up or down, and so project out the spin state $|\uparrow_z\rangle$ or $|\downarrow_z\rangle$. Of course, by rotating the magnet, we can measure the observable $\hat{n}\cdot\vec{\boldsymbol{\sigma}}$ instead.

Our discussion of the quantum eraser has cautioned us that establishing the entangled state eq. (3.8) is *not* sufficient to explain why the measurement procedure prepares an eigenstate of $\mathbf{M}$. In principle, the measurement of the pointer could project out a peculiar superposition of position eigenstates, and so prepare the quantum system in a superposition of $\mathbf{M}$ eigenstates. To achieve a deeper understanding of the measurement process, we will need to explain why the position eigenstate basis of the pointer enjoys a privileged status over other possible bases.

If indeed we can couple any observable to a pointer as just described, and we can observe the pointer, then we can perform any conceivable orthogonal projection in Hilbert space. Given a set of operators $\{\mathbf{E}_a\}$ such that

$$\mathbf{E}_a = \mathbf{E}_a^\dagger, \quad \mathbf{E}_a\mathbf{E}_b = \delta_{ab}\mathbf{E}_a, \quad \sum_a \mathbf{E}_a = \mathbf{1}, \tag{3.11}$$

we can carry out a measurement procedure that will take a pure state $|\psi\rangle\langle\psi|$ to

$$\frac{\mathbf{E}_a|\psi\rangle\langle\psi|\mathbf{E}_a}{\langle\psi|\mathbf{E}_a|\psi\rangle} \tag{3.12}$$

with probability

$$\mathrm{Prob}(a) = \langle\psi|\mathbf{E}_a|\psi\rangle. \tag{3.13}$$

The measurement outcomes can be described by a density matrix obtained by summing over all possible outcomes weighted by the probability of that outcome (rather than by choosing one particular outcome) in which case the measurement modifies the initial pure state according to

$$|\psi\rangle\langle\psi| \to \sum_a \mathbf{E}_a|\psi\rangle\langle\psi|\mathbf{E}_a. \tag{3.14}$$

This is the *ensemble* of pure states describing the measurement outcomes – it is the description we would use if we knew a measurement had been performed, but we did not know the result. Hence, the initial pure state has become a mixed state unless the initial state happened to be an eigenstate of the observable being measured. If the initial state before the measurement were a mixed state with density matrix $\boldsymbol{\rho}$, then by expressing $\boldsymbol{\rho}$ as an ensemble of pure states we find that the effect of the measurement is

$$\boldsymbol{\rho} \to \sum_a \mathbf{E}_a \boldsymbol{\rho} \mathbf{E}_a. \tag{3.15}$$

3.1.2 Generalized measurement

We would now like to generalize the measurement concept beyond these orthogonal measurements considered by Von Neumann. One way to arrive at the idea of a generalized measurement is to suppose that our system A is extended to a tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$, and that we perform orthogonal measurements in the tensor product, which will not necessarily be orthogonal measurements in A alone. At first we will follow a somewhat different course that, while not as well motivated physically, is simpler and more natural from a mathematical view point.

We will suppose that our Hilbert space $\mathcal{H}_A$ is part of a larger space that has the structure of a *direct sum*

$$\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_A^{\perp}. \tag{3.16}$$

Our observers who "live" in $\mathcal{H}_A$ have access only to observables with support in $\mathcal{H}_A$, observables $\mathbf{M}_A$ such that

$$\mathbf{M}_A|\psi^{\perp}\rangle = 0 = \langle\psi^{\perp}|\mathbf{M}_A, \tag{3.17}$$

for any $|\psi^{\perp}\rangle \in \mathcal{H}_A^{\perp}$. For example, in a two-qubit world, we might imagine that our observables have support only when the second qubit is in the state $|0\rangle_2$. Then $\mathcal{H}_A = \mathcal{H}_1 \otimes |0\rangle_2$ and $\mathcal{H}_A^{\perp} = \mathcal{H}_1 \otimes |1\rangle_2$, where $\mathcal{H}_1$ is the Hilbert space of qubit 1. (This situation may seem a bit artificial, which is what I meant in saying that the direct sum decomposition is not so well motivated.) Anyway, when we perform orthogonal measurement in $\mathcal{H}$, preparing one of a set of mutually orthogonal states, our observer will know only about the component of that state in his space $\mathcal{H}_A$. Since these components are not

necessarily orthogonal in $\mathcal{H}_A$, he will conclude that the measurement prepares one of a set or non-orthogonal states.

Let $\{|i\rangle\}$ denote a basis for $\mathcal{H}_A$ and $\{|\mu\rangle\}$ a basis for $\mathcal{H}_A^{\perp}$. Suppose that the initial density matrix $\boldsymbol{\rho}_A$ has support in $\mathcal{H}_A$, and that we perform an orthogonal measurement in $\mathcal{H}$. We will consider the case in which each $\mathbf{E}_a$ is a one-dimensional projector, which will be general enough for our purposes. Thus, $\mathbf{E}_a = |u_a\rangle\langle u_a|$, where $|u_a\rangle$ is a normalized vector in $\mathcal{H}$. This vector has a unique orthogonal decomposition

$$|u_a\rangle = |\tilde{\psi}_a\rangle + |\tilde{\psi}_a^{\perp}\rangle, \tag{3.18}$$

where $|\tilde{\psi}_a\rangle$ and $|\tilde{\psi}_a^{\perp}\rangle$ are (unnormalized) vectors in $\mathcal{H}_A$ and $\mathcal{H}_A^{\perp}$ respectively. After the measurement, the new density matrix will be $|u_a\rangle\langle u_a|$ with probability $\langle u_a|\boldsymbol{\rho}_A|u_a\rangle = \langle\tilde{\psi}_a|\boldsymbol{\rho}_A|\tilde{\psi}_a\rangle$ (since $\boldsymbol{\rho}_A$ has no support on $\mathcal{H}_A^{\perp}$).

But to our observer who knows nothing of $\mathcal{H}_A^{\perp}$, there is no physical distinction between $|u_a\rangle$ and $|\tilde{\psi}_a\rangle$ (aside from normalization). If we write $|\tilde{\psi}_a\rangle = \sqrt{\lambda_a}|\psi_a\rangle$, where $|\psi_a\rangle$ is a normalized state, then for the observer limited to observations in $\mathcal{H}_A$, we might as well say that the outcome of the measurement is $|\psi_a\rangle\langle\psi_a|$ with probability $\langle\tilde{\psi}_a|\boldsymbol{\rho}_A|\tilde{\psi}_a\rangle$.

Let us define an operator

$$\mathbf{F}_a = \mathbf{E}_A \mathbf{E}_a \mathbf{E}_A = |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a| = \lambda_a|\psi_a\rangle\langle\psi_a|, \tag{3.19}$$

(where $\mathbf{E}_A$ is the orthogonal projection taking $\mathcal{H}$ to $\mathcal{H}_A$). Then we may say that the outcome a has probability tr $\mathbf{F}_a\boldsymbol{\rho}$. It is evident that each $\mathbf{F}_a$ is hermitian and nonnegative, but the $\boldsymbol{F}_a$'s are not projections unless $\lambda_a = 1$. Furthermore

$$\sum_a \boldsymbol{F}_a = \boldsymbol{E}_A\left(\sum_a \boldsymbol{E}_a\right)\boldsymbol{E}_A = \boldsymbol{E}_A = \mathbf{1}_A; \tag{3.20}$$

the $\boldsymbol{F}_a$'s sum to the identity on $\mathcal{H}_A$

A partition of unity by nonnegative operators is called a *positive operator-valued measure* (POVM). (The term measure is a bit heavy-handed in our finite-dimensional context; it becomes more apt when the index a can be continually varying.) In our discussion we have arrived at the special case of a POVM by one-dimensional operators (operators with one nonvanishing eigenvalue). In the generalized measurement theory, each outcome has a probability that can be expressed as

$$\text{Prob}(a) = \text{tr } \boldsymbol{\rho}\boldsymbol{F}_a. \tag{3.21}$$

The positivity of $\boldsymbol{F}_a$ is necessary to ensure that the probabilities are positive, and $\sum_a \boldsymbol{F}_a = \mathbf{1}$ ensures that the probabilities sum to unity.

How does a general POVM affect the quantum state? There is not any succinct general answer to this question that is particularly useful, but in the case of a POVM by one-dimensional operators (as just discussed), where the outcome $|\psi_a\rangle\langle\psi_a|$ occurs with probability $\mathrm{tr}(\boldsymbol{F}_a\rho)$, summing over the outcomes yields

$$\begin{aligned}\boldsymbol{\rho} \to \boldsymbol{\rho}' &= \sum_a |\psi_a\rangle\langle\psi_a|(\lambda_a\langle\psi_a|\boldsymbol{\rho}|\psi_a\rangle)\\ &= \sum_a \left(\sqrt{\lambda_a}|\psi_a\rangle\langle\psi_a|\right)\boldsymbol{\rho}\left(\sqrt{\lambda_a}|\psi_a\rangle\langle\psi_a|\right)\\ &= \sum_a \sqrt{\boldsymbol{F}_a}\boldsymbol{\rho}\sqrt{\boldsymbol{F}_a},\end{aligned} \tag{3.22}$$

(which generalizes Von Neumann's $\sum_a \boldsymbol{E}_a\boldsymbol{\rho}\boldsymbol{E}_a$ to the case where the $\boldsymbol{F}_a$'s are not projectors). Note that $\mathrm{tr}\boldsymbol{\rho}' = \mathrm{tr}\boldsymbol{\rho} = 1$ because $\sum_a \boldsymbol{F}_a = 1$.

3.1.3 One-qubit POVM

For example, consider a single qubit and suppose that $\{\hat{n}_a\}$ are N unit 3-vectors that satisfy

$$\sum_a \lambda_a \hat{n}_a = 0, \tag{3.23}$$

where the λ_a's are positive real numbers, $0 < \lambda_a < 1$, such that $\sum_a \lambda_a = 1$. Let

$$\boldsymbol{F}_a = \lambda_a(\mathbf{1} + \hat{n}_a \cdot \vec{\boldsymbol{\sigma}}) = 2\lambda_a \boldsymbol{E}(\hat{n}_a), \tag{3.24}$$

(where $\boldsymbol{E}(\hat{n}_a)$ is the projection $|\uparrow_{\hat{n}_a}\rangle\langle\uparrow_{\hat{n}_a}|$). Then

$$\sum_a \boldsymbol{F}_a = (\sum_a \lambda_a)\mathbf{1} + (\sum_a \lambda_a\hat{n}_a)\cdot\vec{\boldsymbol{\sigma}} = \mathbf{1}; \tag{3.25}$$

hence the $\boldsymbol{F}$'s define a POVM.

In the case $N = 2$, we have $\hat{n}_1 + \hat{n}_2 = 0$, so our POVM is just an orthogonal measurement along the $\hat{n}_1$ axis. For $N = 3$, in the symmetric case $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$. We have $\hat{n}_1 + \hat{n}_2 + \hat{n}_3 = 0$, and

$$\boldsymbol{F}_a = \frac{1}{3}(\mathbf{1} + \hat{n}_a \cdot \vec{\boldsymbol{\sigma}}) = \frac{2}{3}\boldsymbol{E}(\hat{n}_a). \tag{3.26}$$

3.1.4 Neumark's theorem

We arrived at the concept of a POVM by considering orthogonal measurement in a space larger than $\mathcal{H}_A$. Now we will reverse our tracks, showing that any POVM can be realized in this way.

So consider an arbitrary POVM with n one-dimensional positive operators $\boldsymbol{F}_a$ satisfying $\sum_{a=1}^n \boldsymbol{F}_a = \mathbf{1}$. We will show that this POVM can always be realized by extending the Hilbert space to a larger space, and performing orthogonal measurement in the larger space. This statement is called *Neumark's theorem.*[1]

To prove it, consider a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} = N$, and a POVM $\{\boldsymbol{F}_a\}, a = 1, \ldots, n$, with $n \geq N$. Each one-dimensional positive operator can be written

$$\boldsymbol{F}_a = |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a|, \tag{3.27}$$

where the vector $|\tilde{\psi}_a\rangle$ is not normalized. Writing out the matrix elements explicitly, the property $\sum_a \boldsymbol{F}_a = \mathbf{1}$ becomes

$$\sum_{a=1}^n (F_a)_{ij} = \sum_{a=1}^n \tilde{\psi}^*_{ai}\tilde{\psi}_{aj} = \delta_{ij}. \tag{3.28}$$

Now let's change our perspective on eq. (3.28). Interpret the $(\psi_a)_i$'s not as $n \geq N$ vectors in an N-dimensional space, but rather an $N \leq n$ vectors $(\psi_i^T)_a$ in an n-dimensional space. Then eq. (3.28) becomes the statement that these N vectors form an orthonormal set. Naturally, it is possible to extend these vectors to an orthonormal basis for an n-dimensional space. In other words, there is an $n \times n$ matrix u_{ai}, with $u_{ai} = \tilde{\psi}_{ai}$ for $i = 1, 2, \ldots, N$, such that

$$\sum_a u^*_{ai}u_{aj} = \delta_{ij}, \tag{3.29}$$

or, in matrix form $\boldsymbol{U}^\dagger\boldsymbol{U} = \mathbf{1}$. It follows that $\boldsymbol{U}\boldsymbol{U}^\dagger = \mathbf{1}$, since

$$\boldsymbol{U}(\boldsymbol{U}^\dagger\boldsymbol{U})|\psi\rangle = (\boldsymbol{U}\boldsymbol{U}^\dagger)\boldsymbol{U}|\psi\rangle = \boldsymbol{U}|\psi\rangle \tag{3.30}$$

[1] For a discussion of POVM's and Neumark's theorem, see A. Peres, *Quantum Theory: Concepts and Methods.*

for any vector $|\psi\rangle$, and (at least for finite-dimensional matrices) the range of $\boldsymbol{U}$ is the whole n-dimension space. Returning to the component notation, we have

$$\sum_j u_{aj} u_{bj}^* = \delta_{ab}, \tag{3.31}$$

so the $(u_a)_i$ are a set of n orthonormal vectors.[2]

Now suppose that we perform an orthogonal measurement in the space of dimension $n \geq N$ defined by

$$\boldsymbol{E}_a = |u_a\rangle\langle u_a|. \tag{3.32}$$

We have constructed the $|u_a\rangle$'s so that each has an orthogonal decomposition

$$|u_a\rangle = |\tilde{\psi}_a\rangle + |\tilde{\psi}_a^\perp\rangle; \tag{3.33}$$

where $|\tilde{\psi}_a\rangle \in \mathcal{H}$ and $|\tilde{\psi}_a^\perp\rangle \in \mathcal{H}^\perp$. By orthogonally projecting this basis onto $\mathcal{H}$, then, we recover the POVM $\{\boldsymbol{F}_a\}$. This completes the proof of Neumark's theorem.

To illustrate Neumark's theorem in action, consider again the POVM on a single qubit with

$$\boldsymbol{F}_a = \frac{2}{3}|\uparrow_{\hat{n}_a}\rangle\langle\uparrow_{\hat{n}_a}|, \tag{3.34}$$

$a = 1, 2, 3$, where $0 = \hat{n}_1 + \hat{n}_2 + \hat{n}_3$. According to the theorem, this POVM can be realized as an orthogonal measurement on a "qutrit," a quantum system in a three-dimensional Hilbert space.

Let $\hat{n}_1 = (0, 0, 1), \hat{n}_2 = (\sqrt{3}/2, 0, -1/2), \hat{n}_3 = (-\sqrt{3}/2, 0, 0, -1/2)$, and therefore, recalling that

$$|\theta, \varphi = 0\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \tag{3.35}$$

we may write the three vectors $|\tilde{\psi}_a\rangle = \sqrt{2/3}|\theta_a, \varphi = 0\rangle$ (where $\theta_1, \theta_2, \theta_3 = 0, 2\pi/3, 4\pi/3$) as

$$|\tilde{\psi}_1\rangle, |\tilde{\psi}_2\rangle, |\tilde{\psi}_3\rangle = \begin{pmatrix} \sqrt{2/3} \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1/6} \\ \sqrt{1/2} \end{pmatrix}, \begin{pmatrix} -\sqrt{1/6} \\ \sqrt{1/2} \end{pmatrix}. \tag{3.36}$$

[2] In other words, we have shown that if the rows of an $n \times n$ matrix are orthonormal, then so are the columns.

Now, we may interpret these three two-dimensional vectors as a 2×3 matrix, and as Neumark's theorem assured us, the two rows are orthonormal. Hence we can add one more orthonormal row:

$$|u_1\rangle, |u_2\rangle, |u_3\rangle = \begin{pmatrix} \sqrt{2/3} \\ 0 \\ \sqrt{1/3} \end{pmatrix}, \begin{pmatrix} \sqrt{1/6} \\ \sqrt{1/2} \\ -\sqrt{1/3} \end{pmatrix}, \begin{pmatrix} -\sqrt{1/6} \\ \sqrt{1/2} \\ \sqrt{1/3} \end{pmatrix}, \tag{3.37}$$

and we see (as the theorem also assured us) that the columns (the $|u_a\rangle$'s) are then orthonormal as well. If we perform an orthogonal measurement onto the $|u_a\rangle$ basis, an observer cognizant of only the two-dimensional subspace will conclude that we have performed the POVM $\{\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3\}$. We have shown that if our qubit is secretly two components of a *qutrit*, the POVM may be realized as orthogonal measurement of the qutrit.

3.1.5 Orthogonal measurement on a tensor product

A typical qubit harbors no such secret, though. To perform a generalized measurement, we will need to provide additional qubits, and perform joint orthogonal measurements on several qubits at once.

So *now* we consider the case of two (isolated) systems A and B, described by the tensor product $\mathcal{H}_A \oplus \mathcal{H}_B$. Suppose we perform an orthogonal measurement on the tensor product, with

$$\sum_a \boldsymbol{E}_a = \boldsymbol{1}, \tag{3.38}$$

where all $\boldsymbol{E}_a$'s are mutually orthogonal projectors. Let us imagine that the initial system of the quantum system is an "uncorrelated" tensor product state

$$\boldsymbol{\rho}_{AB} = \boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B. \tag{3.39}$$

Then outcome a occurs with probability

$$\text{Prob}(a) = \text{tr}_{AB}[\boldsymbol{E}_a(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B)], \tag{3.40}$$

in which case the new density matrix will be

$$\rho'_{AB}(a) = \frac{\boldsymbol{E}_a(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B)\boldsymbol{E}_a}{\text{tr}_{AB}[\boldsymbol{E}_a(\rho_A \otimes \rho_B)]}. \tag{3.41}$$

To an observer who has access only to system A, the new density matrix for that system is given by the partial trace of the above, or

$$\boldsymbol{\rho}'_A(a) = \frac{\mathrm{tr}_B[\boldsymbol{E}_a(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B)\boldsymbol{E}_a]}{\mathrm{tr}_{AB}[\boldsymbol{E}_a(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B)]}. \tag{3.42}$$

The expression eq. (3.40) for the probability of outcome a can also be written

$$\mathrm{Prob}(a) = \mathrm{tr}_A[\mathrm{tr}_B(\boldsymbol{E}_a(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B))] = \mathrm{tr}_A(\boldsymbol{F}_a\boldsymbol{\rho}_A); \tag{3.43}$$

If we introduce orthonormal bases $\{|i\rangle_A\}$ for $\mathcal{H}_A$ and $|\mu\rangle_B$ for $\mathcal{H}_B$, then

$$\sum_{ij\mu\nu}(E_a)_{j\nu,i\mu}(\rho_A)_{ij}(\rho_B)_{\mu\nu} = \sum_{ij}(F_a)_{ji}(\rho_A)_{ij}, \tag{3.44}$$

or

$$(F_a)_{ji} = \sum_{\mu\nu}(E_a)_{j\nu,i\mu}(\rho_B)_{\mu\nu}. \tag{3.45}$$

It follows from eq. (3.45) that each $\boldsymbol{F}_a$ has the properties:

(1) Hermiticity:

$$\begin{aligned}(F_a)^*_{ij} &= \sum_{\mu\nu}(E_a)^*_{i\nu,j\mu}(\rho_B)^*_{\mu\nu}\\ &= \sum_{\mu\nu}(E_a)_{j\mu,i\nu}(\rho_B)_{\nu\mu} = F_{ji}\end{aligned}$$

(because $\boldsymbol{E}_a$ and $\boldsymbol{\rho}_B$ are hermitian.

(2) Positivity:

In the basis that diagonalizes $\boldsymbol{\rho}_B = \sum_\mu p_\mu |\mu\rangle_B\,{}_B\langle\mu|$, ${}_A\langle\psi|\boldsymbol{F}_a|\psi\rangle_A = \sum_\mu p_\mu ({}_A\langle\psi| \otimes {}_B\langle\mu|)\boldsymbol{E}_a(|\psi\rangle_A \otimes |\mu\rangle_B)$

≥ 0 (because $\boldsymbol{E}_a$ is positive).

(3) Completeness:

$$\sum_a \boldsymbol{F}_a = \sum_\mu p_\mu\ {}_B\langle\mu|\sum_a \boldsymbol{E}_a|\mu\rangle_B = \mathbf{1}_A$$

(because $\sum_a \boldsymbol{E}_a = \mathbf{1}_{AB}$ and tr $\boldsymbol{\rho}_B = 1$).

But the $\boldsymbol{F}_a$'s need not be mutually orthogonal. In fact, the number of $\boldsymbol{F}_a$'s is limited only by the dimension of $\mathcal{H}_A \otimes \mathcal{H}_B$, which is greater than (and perhaps *much* greater than) the dimension of $\mathcal{H}_A$.

There is no simple way, in general, to express the final density matrix $\boldsymbol{\rho}'_A(a)$ in terms of $\boldsymbol{\rho}_A$ and $\boldsymbol{F}_a$. But let us disregard how the POVM changes the density matrix, and instead address this question: Suppose that $\mathcal{H}_A$ has dimension N, and consider a POVM with n one-dimensional nonnegative $\boldsymbol{F}_a$'s satisfying $\sum_{a=1}^n \boldsymbol{F}_a = \mathbf{1}_A$. Can we choose the space $\mathcal{H}_B$, density matrix $\boldsymbol{\rho}_B$ in $\mathcal{H}_B$, and projection operators $\boldsymbol{E}_a$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ (where the number or $\boldsymbol{E}_a$'s may exceed the number of $\boldsymbol{F}_a$'s) such that the probability of outcome a of the orthogonal measurement satisfies[3]

$$\mathrm{tr}\ \boldsymbol{E}_a(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B) = \mathrm{tr}(\boldsymbol{F}_a\boldsymbol{\rho}_A)\ ? \tag{3.46}$$

(Never mind how the orthogonal projections modify $\boldsymbol{\rho}_A$!) We will consider this to be a "realization" of the POVM by orthogonal measurement, because we have no interest in what the state $\boldsymbol{\rho}'_A$ is for each measurement outcome; we are only asking that the *probabilities* of the outcomes agree with those defined by the POVM.

Such a realization of the POVM is indeed possible; to show this, we will appeal once more to Neumark's theorem. Each one-dimensional $\boldsymbol{F}_a$, $a = 1, 2, \ldots, n$, can be expressed as $\boldsymbol{F}_a = |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a|$. According to Neumark, there are n orthonormal n-component vectors $|u_a\rangle$ such that

$$|u_a\rangle = |\tilde{\psi}_a\rangle + |\tilde{\psi}_a^\perp\rangle. \tag{3.47}$$

Now consider, to start with, the special case $n = rN$, where r is a positive integer. Then it is convenient to decompose $|\tilde{\psi}_a^\perp\rangle$ as a direct sum of $r-1$ N-component vectors:

$$|\tilde{\psi}_a^\perp\rangle = |\tilde{\psi}_{1,a}^\perp\rangle \oplus |\tilde{\psi}_{2,a}^\perp\rangle \oplus \cdots \oplus |\tilde{\psi}_{r-1,a}^\perp\rangle; \tag{3.48}$$

Here $|\tilde{\psi}_{1,a}^\perp\rangle$ denotes the first N components of $|\tilde{\psi}_a^\perp\rangle$, $|\tilde{\psi}_{2,a}^\perp\rangle$ denotes the next N components, etc. Then the orthonormality of the $|u_a\rangle$'s implies that

$$\delta_{ab} = \langle u_a|u_b\rangle = \langle\tilde{\psi}_a|\tilde{\psi}_b\rangle + \sum_{\mu=1}^{r-1}\langle\tilde{\psi}_{\mu,a}^\perp|\tilde{\psi}_{\mu,b}^\perp\rangle\ . \tag{3.49}$$

[3] If there are more $\boldsymbol{E}_a$'s than $\boldsymbol{F}_a$'s, all but n outcomes have probability *zero*.

Now we will choose $\mathcal{H}_B$ to have dimension r and we will denote an orthonormal basis for $\mathcal{H}_B$ by

$$\{|\mu\rangle_B\}, \quad \mu = 0, 1, 2, \ldots, r-1. \tag{3.50}$$

Then it follows from Eq. (3.49) that

$$|\Phi_a\rangle_{AB} = |\tilde{\psi}_a\rangle_A|0\rangle_B + \sum_{\mu=1}^{r-1} |\tilde{\psi}^{\perp}_{\mu,a}\rangle_A|\mu\rangle_B, \quad a = 1, 2, \ldots, n, \tag{3.51}$$

is an orthonormal basis for $\mathcal{H}_A \otimes \mathcal{H}_B$.

Now suppose that the state in $\mathcal{H}_A \otimes \mathcal{H}_B$ is

$$\boldsymbol{\rho}_{AB} = \boldsymbol{\rho}_A \otimes |0\rangle_B\, {}_B\langle 0|, \tag{3.52}$$

and that we perform an orthogonal projection onto the basis $\{|\Phi_a\rangle_{AB}\}$ in $\mathcal{H}_A \otimes \mathcal{H}_B$. Then, since ${}_B\langle 0|\mu\rangle_B = 0$ for $\mu \neq 0$, the outcome $|\Phi_a\rangle_{AB}$ occurs with probability

$${}_{AB}\langle\Phi_a|\boldsymbol{\rho}_{AB}|\Phi_a\rangle_{AB} = {}_A\langle\tilde{\psi}_a|\boldsymbol{\rho}_A|\tilde{\psi}_a\rangle_A\ , \tag{3.53}$$

and thus,

$$\langle\Phi_a|\boldsymbol{\rho}_{AB}|\Phi_a\rangle_{AB} = \mathrm{tr}(\boldsymbol{F}_a\boldsymbol{\rho}_A). \tag{3.54}$$

We have indeed succeeded in "realizing" the POVM $\{\boldsymbol{F}_a\}$ by performing orthogonal measurement on $\mathcal{H}_A \otimes \mathcal{H}_B$. This construction is just as efficient as the "direct sum" construction described previously; we performed orthogonal measurement in a space of dimension $n = N \cdot r$.

If outcome a occurs, then the state

$$\boldsymbol{\rho}'_{AB} = |\Phi_a\rangle_{AB}\, {}_{AB}\langle\Phi_a|, \tag{3.55}$$

is prepared by the measurement. The density matrix seen by an observer who can probe only system A is obtained by performing a partial trace over $\mathcal{H}_B$,

$$\begin{aligned}\boldsymbol{\rho}'_A &= \mathrm{tr}_B\left(|\Phi_a\rangle_{AB}\, {}_{AB}\langle\Phi_a|\right)\\ &= |\tilde{\psi}_a\rangle_A\, {}_A\langle\tilde{\psi}_a| + \sum_{\mu=1}^{r-1} |\tilde{\psi}^{\perp}_{\mu,a}\rangle_A\, {}_A\langle\tilde{\psi}^{\perp}_{\mu,a}|\end{aligned} \tag{3.56}$$

which isn't quite the same thing as what we obtained in our "direct sum" construction. In any case, there are many possible ways to realize a POVM by orthogonal measurement and eq. (3.56) applies only to the particular construction we have chosen here.

Nevertheless, this construction really is perfectly adequate for realizing the POVM in which the state $|\psi_a\rangle_A\, {}_A\langle\psi_a|$ is prepared in the event that outcome a occurs. The hard part of implementing a POVM is assuring that outcome a arises with the desired probability. It is then easy to arrange that the *result* in the event of outcome a is the state $|\psi_a\rangle_A\, {}_A\langle\psi_a|$; if we like, once the measurement is performed and outcome a is found, we can simply throw $\boldsymbol{\rho}_A$ away and proceed to prepare the desired state! In fact, in the case of the projection onto the basis $|\Phi_a\rangle_{AB}$, we can complete the construction of the POVM by projecting system B onto the $\{|\mu\rangle_B\}$ basis, and communicating the result to system A. If the outcome is $|0\rangle_B$, then no action need be taken. If the outcome is $|\mu\rangle_B$, $\mu > 0$, then the state $|\tilde{\psi}^{\perp}_{\mu,a}\rangle_A$ has been prepared, which can then be rotated to $|\psi_a\rangle_A$.

So far, we have discussed only the special case $n = rN$. But if actually $n = rN - c$, $0 < c < N$, then we need only choose the final c components of $|\tilde{\psi}^{\perp}_{r-1,a}\rangle_A$ to be zero, and the states $|\Phi\rangle_{AB}$ will still be mutually orthogonal. To complete the orthonormal basis, we may add the c states

$$|e_i\rangle_A|r-1\rangle_B, \quad i = N-c+1, N-c+2, \ldots N\ ; \tag{3.57}$$

here e_i is a vector whose only nonvanishing component is the ith component, so that $|e_i\rangle_A$ is guaranteed to be orthogonal to $|\tilde{\psi}^{\perp}_{r-1,a}\rangle_A$. In this case, the POVM is realized as an orthogonal measurement on a space of dimension $rN = n + c$.

As an example of the tensor product construction, we may consider once again the single-qubit POVM with

$$\boldsymbol{F}_a = \frac{2}{3}|\uparrow_{\hat{n}_a}\rangle_A\, {}_A\langle\uparrow_{\hat{n}_a}|, \quad a = 1, 2, 3. \tag{3.58}$$

We may realize this POVM by introducing a second qubit B. In the two-qubit Hilbert space, we may project onto the orthonormal basis[4]

$$
\begin{aligned}
|\Phi_a\rangle &= \sqrt{\frac{2}{3}}|\uparrow_{\hat{n}_a}\rangle_A|0\rangle_B + \sqrt{\frac{1}{3}}|0\rangle_A|1\rangle_B, \quad a = 1,2,3,\\
|\Phi_0\rangle &= |1\rangle_A|1\rangle_B.
\end{aligned} \tag{3.59}
$$

If the initial state is $\boldsymbol{\rho}_{AB} = \boldsymbol{\rho}_A \otimes |0\rangle_B\ {}_B\langle 0|$, we have

$$
\langle\Phi_a|\boldsymbol{\rho}_{AB}|\Phi_a\rangle = \frac{2}{3}{}_A\langle\uparrow_{\hat{n}_a}|\boldsymbol{\rho}_A|\uparrow_{\hat{n}_a}\rangle_A \tag{3.60}
$$

so this projection implements the POVM on $\mathcal{H}_A$. (This time we performed orthogonal measurements in a four-dimensional space; we only needed three dimensions in our earlier "direct sum" construction.)

3.1.6 GHJW with POVM's

In our discussion of the GHJW theorem, we saw that by preparing a state

$$
|\Phi\rangle_{AB} = \sum_\mu \sqrt{q_\mu}|\psi_\mu\rangle_A|\beta_\mu\rangle_B, \tag{3.61}
$$

we can realize the ensemble

$$
\boldsymbol{\rho}_A = \sum_\mu q_\mu |\psi_\mu\rangle_A\ {}_A\langle\psi_\mu|, \tag{3.62}
$$

by performing orthogonal measurements on $\mathcal{H}_B$. Moreover, if $\dim\mathcal{H}_B = n$, then for this single pure state $|\Phi\rangle_{AB}$, we can realize any preparation of $\boldsymbol{\rho}_A$ as an ensemble of up to n pure states by measuring an appropriate observable on $\mathcal{H}_B$.

But we can now see that if we are willing to allow POVM's on $\mathcal{H}_B$ rather than orthogonal measurements only, then even for $\dim\mathcal{H}_B = N$, we can realize *any* preparation of $\boldsymbol{\rho}_A$ by choosing the POVM on $\mathcal{H}_B$ appropriately. The point is that $\boldsymbol{\rho}_B$ has support on a space that is at most dimension N. We may therefore rewrite $|\Phi\rangle_{AB}$ as

$$
|\Phi\rangle_{AB} = \sum_\mu \sqrt{q_\mu}|\psi_\mu\rangle_A|\tilde{\beta}_\mu\rangle_B, \tag{3.63}
$$

[4] Here the phase of $|\tilde{\psi}_2\rangle = \sqrt{2/3}|\uparrow_{\hat{n}_2}\rangle$ differs by -1 from that in eq. (3.36); it has been chosen so that $\langle\uparrow_{\hat{n}_a}|\uparrow_{\hat{n}_b}\rangle = -1/2$ for $a \neq b$. We have made this choice so that the coefficient of $|0\rangle_A|1\rangle_B$ is positive in all three of $|\Phi_1\rangle, |\Phi_2\rangle, |\Phi_3\rangle$.

where $|\tilde{\beta}_\mu\rangle_B$ is the result of orthogonally projecting $|\beta_\mu\rangle_B$ onto the support of $\boldsymbol{\rho}_B$. We may now perform the POVM on the support of $\boldsymbol{\rho}_B$ with $\boldsymbol{F}_\mu = |\tilde{\beta}_\mu\rangle_B\ {}_B\langle\tilde{\beta}_\mu|$, and thus prepare the state $|\psi_\mu\rangle_A$ with probability q_μ.

3.2 Superoperators

3.2.1 The operator-sum representation

We now proceed to the next step of our program of understanding the behavior of one part of a bipartite quantum system. We have seen that a pure state of the bipartite system may behave like a mixed state when we observe subsystem A alone, and that an orthogonal measurement of the bipartite system may be a (nonorthogonal) POVM on A alone. Next we ask, if a state of the bipartite system undergoes unitary evolution, how do we describe the evolution of A alone?

Suppose that the initial density matrix of the bipartite system is a tensor product state of the form

$$
\boldsymbol{\rho}_A \otimes |0\rangle_B\ {}_B\langle 0|; \tag{3.64}
$$

system A has density matrix $\boldsymbol{\rho}_A$, and system B is assumed to be in a pure state that we have designated $|0\rangle_B$. The bipartite system evolves for a finite time, governed by the unitary time evolution operator

$$
\mathbf{U}_{AB}\left(\boldsymbol{\rho}_A \otimes |0\rangle_B\ {}_B\langle 0|\right)\mathbf{U}_{AB}. \tag{3.65}
$$

Now we perform the partial trace over $\mathcal{H}_B$ to find the final density matrix of system A,

$$
\begin{aligned}
\boldsymbol{\rho}'_A &= \mathrm{tr}_B\left(\mathbf{U}_{AB}\left(\boldsymbol{\rho}_A \otimes |0\rangle_B\ {}_B\langle 0|\right)\mathbf{U}^\dagger_{AB}\right)\\
&= \sum_\mu {}_B\langle\mu|\mathbf{U}_{AB}|0\rangle_B\boldsymbol{\rho}_A\ {}_B\langle 0|\mathbf{U}_{AB}|\mu\rangle_B,
\end{aligned} \tag{3.66}
$$

where $\{|\mu\rangle_B\}$ is an orthonormal basis for $\mathcal{H}_{B'}$ and ${}_B\langle\mu|\mathbf{U}_{AB}|0\rangle_B$ is an operator acting on $\mathcal{H}_A$. (If $\{|i\rangle_A \otimes |\mu\rangle_B\}$ is an orthonormal basis for $\mathcal{H}_A \otimes \mathcal{H}_B$, then ${}_B\langle\mu|\mathbf{U}_{AB}|\nu\rangle_B$ denotes the operator whose matrix elements are

$$
{}_A\langle i|\left({}_B\langle\mu|\mathbf{U}_{AB}|\nu\rangle_B\right)|j\rangle_A
$$

$$= \left({}_A\langle i| \otimes {}_B\langle \mu|\right) \mathbf{U}_{AB} \left(|j\rangle_A \otimes |\nu\rangle_B\right).) \tag{3.67}$$

If we denote

$$\mathbf{M}_\mu = {}_B\langle \mu|\mathbf{U}_{AB}|0\rangle_B, \tag{3.68}$$

then we may express $\boldsymbol{\rho}'_A$ as

$$\$(\boldsymbol{\rho}_A) \equiv \boldsymbol{\rho}'_A = \sum_\mu \mathbf{M}_\mu \boldsymbol{\rho}_A \mathbf{M}_\mu^\dagger. \tag{3.69}$$

It follows from the unitarity of $\mathbf{U}_{AB}$ that the $\mathbf{M}_\mu$'s satisfy the property

$$\begin{aligned} \sum_\mu \mathbf{M}_\mu^\dagger \mathbf{M}_\mu &= \sum_\mu {}_B\langle 0|\mathbf{U}_{AB}^\dagger|\mu\rangle_B \; {}_B\langle \mu|\mathbf{U}_{AB}|0\rangle_B \\ &= {}_B\langle 0|\mathbf{U}_{AB}^\dagger \mathbf{U}_{AB}|0\rangle_B = \mathbf{1}_A. \end{aligned} \tag{3.70}$$

Eq. (3.69) defines a linear map \$ that takes linear operators to linear operators. Such a map, if the property in eq. (3.70) is satisfied, is called a *superoperator*, and eq. (3.69) is called the operator sum representation (or *Kraus* representation) of the superoperator. A superoperator can be regarded as a linear map that takes density operators to density operators, because it follows from eq. (3.69) and eq. (3.70) that $\boldsymbol{\rho}'_A$ is a density matrix if $\boldsymbol{\rho}_A$ is:

(1) $\boldsymbol{\rho}'_A$ is hermitian: $\boldsymbol{\rho}'^\dagger_A = \sum_\mu \mathbf{M}_\mu \boldsymbol{\rho}_A^\dagger \mathbf{M}_\mu^\dagger = \boldsymbol{\rho}_A$.

(2) $\boldsymbol{\rho}'_A$ has unit trace: $\mathrm{tr}\boldsymbol{\rho}'_A = \sum_\mu \mathrm{tr}(\boldsymbol{\rho}_A \mathbf{M}_\mu^\dagger \mathbf{M}_\mu) = \mathrm{tr}\boldsymbol{\rho}_A = 1$.

(3) $\boldsymbol{\rho}'_A$ is positive: ${}_A\langle\psi|\boldsymbol{\rho}'_A|\psi\rangle_A = \sum_\mu (\langle\psi|\mathbf{M}_\mu)\boldsymbol{\rho}_A(\mathbf{M}_\mu^\dagger|\psi\rangle) \geq 0$.

We showed that the operator sum representation in eq. (3.69) follows from the "unitary representation" in eq. (3.66). But furthermore, given the operator sum representation of a superoperator, it is always possible to construct a corresponding unitary representation. We choose $\mathcal{H}_B$ to be a Hilbert space whose dimension is at least as large as the number of terms in the operator sum. If $\{|\varphi_A\}$ is any vector in $\mathcal{H}_A$, the $\{|\mu\rangle_B\}$ are orthonormal states in $\mathcal{H}_B$, and $|0\rangle_B$ is some normalized state in $\mathcal{H}_B$, define the action of $\mathbf{U}_{AB}$ by

$$\mathbf{U}_{AB}\left(|\varphi\rangle_A \otimes |0\rangle_B\right) = \sum_\mu \mathbf{M}_\mu |\varphi\rangle_A \otimes |\mu\rangle_B. \tag{3.71}$$

This action is inner product preserving:

$$\begin{aligned} &\left(\sum_\nu {}_A\langle\varphi_2|\mathbf{M}_\nu^\dagger \otimes {}_B\langle\nu|\right)\left(\sum_\mu \mathbf{M}_\mu|\varphi_1\rangle_A \otimes |\mu\rangle_B\right) \\ &= {}_A\langle\varphi_2|\sum_\mu \mathbf{M}_\mu^\dagger \mathbf{M}_\mu|\varphi_1\rangle_A = {}_A\langle\varphi_2|\varphi_1\rangle_A; \end{aligned} \tag{3.72}$$

therefore, $\mathbf{U}_{AB}$ can be extended to a unitary operator acting on all of $\mathcal{H}_A \otimes \mathcal{H}_B$. Taking the partial trace we find

$$\begin{aligned} &\mathrm{tr}_B\left(\mathbf{U}_{AB}\left(|\varphi\rangle_A \otimes |0\rangle_B\right)\left({}_A\langle\varphi| \otimes {}_B\langle 0|\right)\mathbf{U}_{AB}^\dagger\right) \\ &= \sum_\mu \mathbf{M}_\mu \left(|\varphi\rangle_A \, {}_A\langle\varphi|\right)\mathbf{M}_\mu^\dagger. \end{aligned} \tag{3.73}$$

Since any $\boldsymbol{\rho}_A$ can be expressed as an ensemble of pure states, we recover the operator sum representation acting on an arbitrary $\boldsymbol{\rho}_A$.

It is clear that the operator sum representation of a given superoperator \$ is not unique. We can perform the partial trace in any basis we please. If we use the basis $\{{}_B\langle\nu'| = \sum_\mu U_{\nu\mu} \, {}_B\langle\mu|\}$ then we obtain the representation

$$\$(\boldsymbol{\rho}_A) = \sum_\nu \mathbf{N}_\nu \boldsymbol{\rho}_A \mathbf{N}_\nu^\dagger, \tag{3.74}$$

where $\mathbf{N}_\nu = U_{\nu\mu}\mathbf{M}_\mu$. We will see shortly that *any* two operator-sum representations of the same superoperator are always related this way.

Superoperators are important because they provide us with a formalism for discussing the general theory of *decoherence*, the evolution of pure states into mixed states. *Unitary* evolution of $\boldsymbol{\rho}_A$ is the special case in which there is only one term in the operator sum. If there are two or more terms, then there are pure initial states of $\mathcal{H}_A$ that become *entangled* with $\mathcal{H}_B$ under evolution governed by $\mathbf{U}_{AB}$. That is, if the operators M_1 and M_2 appearing in the operator sum are linearly independent, then there is a vector $|\varphi\rangle_A$ such that $|\tilde{\varphi}_1\rangle_A = M_1|\varphi\rangle_A$ and $|\tilde{\varphi}_2\rangle_A = M_2|\varphi\rangle_A$ are linearly independent, so that the state $|\tilde{\varphi}_1\rangle_A|1\rangle_B + |\tilde{\varphi}_2\rangle_A|2\rangle_B + \cdots$ has Schmidt number greater than one. Therefore, the pure state $|\varphi\rangle_A \, {}_A\langle\varphi|$ evolves to the *mixed* final state $\boldsymbol{\rho}'_A$.

Two superoperators $\$_1$ and $\$_2$ can be composed to obtain another superoperator $\$_2 \circ \$_1$; if $\$_1$ describes evolution from yesterday to today, and $\$_2$

describes evolution from today to tomorrow, then $\$_2 \circ \$_1$ describes the evolution from yesterday to tomorrow. But is the inverse of a superoperator also a superoperator; that is, is there a superoperator that describes the evolution from today to yesterday? In fact, you will show in a homework exercise that a superoperator is invertible only if it is unitary.

Unitary evolution operators form a group, but superoperators define a dynamical *semigroup*. When decoherence occurs, there is an arrow of time; even at the microscopic level, one can tell the difference between a movie that runs forwards and one running backwards. Decoherence causes an irrevocable loss of quantum information — once the (dead) cat is out of the bag, we can't put it back in again.

3.2.2 Linearity

Now we will broaden our viewpoint a bit and consider the essential properties that should be satisfied by any "reasonable" time evolution law for density matrices. We will see that any such law admits an operator-sum representation, so in a sense the dynamical behavior we extracted by considering part of a bipartite system is actually the most general possible.

A mapping $\$: \boldsymbol{\rho} \to \boldsymbol{\rho}'$ that takes an initial density matrix $\boldsymbol{\rho}$ to a final density matrix $\boldsymbol{\rho}'$ is a mapping of operators to operators that satisfies

(1) $\$$ *preserves hermiticity*: $\boldsymbol{\rho}'$ hermitian if $\boldsymbol{\rho}$ is.

(2) $\$$ is *trace preserving*: $\mathrm{tr}\boldsymbol{\rho}' = 1$ if $\mathrm{tr}\boldsymbol{\rho} = 1$.

(3) $\$$ is *positive*: $\boldsymbol{\rho}'$ is nonnegative if $\boldsymbol{\rho}$ is.

It is also customary to assume

(0) $\$$ is linear.

While (1), (2), and (3) really are necessary if $\boldsymbol{\rho}'$ is to be a density matrix, (0) is more open to question. Why linearity?

One possible answer is that nonlinear evolution of the density matrix would be hard to reconcile with any ensemble interpretation. If

$$\$\left(\boldsymbol{\rho}(\lambda)\right) \equiv \$\left(\lambda\boldsymbol{\rho}_1 + (1-\lambda)\boldsymbol{\rho}_2\right) = \lambda\$(\boldsymbol{\rho}_1) + (1-\lambda)\$(\boldsymbol{\rho}_2), \tag{3.75}$$

then time evolution is faithful to the probabilistic interpretation of $\boldsymbol{\rho}(\lambda)$: either (with probability λ) $\boldsymbol{\rho}_1$ was initially prepared and evolved to $\$(\boldsymbol{\rho}_1)$, or (with probability $1-\lambda$) $\boldsymbol{\rho}_2$ was initially prepared and evolved to $\$(\boldsymbol{\rho}_2)$. But a nonlinear $\$$ typically has consequences that are seemingly paradoxical.

Consider, for example, a single qubit evolving according to

$$\$(\boldsymbol{\rho}) = \exp\left[i\pi\boldsymbol{\sigma}_1\mathrm{tr}(\boldsymbol{\sigma}_1\boldsymbol{\rho})\right]\ \boldsymbol{\rho}\ \exp\left[-i\pi\boldsymbol{\sigma}_1\mathrm{tr}(\boldsymbol{\sigma}_1\boldsymbol{\rho})\right]\ . \tag{3.76}$$

One can easily check that $\$$ is positive and trace-preserving. Suppose that the initial density matrix is $\boldsymbol{\rho} = \frac{1}{2}\mathbf{1}$, realized as the ensemble

$$\boldsymbol{\rho} = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z|. \tag{3.77}$$

Since $\mathrm{tr}(\boldsymbol{\sigma}_1\boldsymbol{\rho}) = 0$, the evolution of $\boldsymbol{\rho}$ is trivial, and both representatives of the ensemble are unchanged. If the spin was prepared as $|\uparrow_z\rangle$, it remains in the state $|\uparrow_z\rangle$.

But now imagine that, immediately after preparing the ensemble, we do nothing if the state has been prepared as $|\uparrow_z\rangle$, but we rotate it to $|\uparrow_x\rangle$ if it has been prepared as $|\downarrow_z\rangle$. The density matrix is now

$$\boldsymbol{\rho}' = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\uparrow_x\rangle|\uparrow_x\rangle, \tag{3.78}$$

so that $\mathrm{tr}\boldsymbol{\rho}'\boldsymbol{\sigma}_1 = \frac{1}{2}$. Under evolution governed by $\$$, this becomes $\$(\boldsymbol{\rho}') = \boldsymbol{\sigma}_1\boldsymbol{\rho}'\boldsymbol{\sigma}_1$. In this case then, if the spin was prepared as $|\uparrow_z\rangle$, it evolves to the orthogonal state $|\downarrow_z\rangle$.

The state initially prepared as $|\uparrow_z\rangle$ evolves differently under these two scenarios. But what is the difference between the two cases? The difference was that *if* the spin was initially prepared as $|\downarrow_z\rangle$, we took different actions: doing nothing in case (1) but rotating the spin in case (2). Yet we have found that the spin behaves differently in the two cases, even if it was initially prepared as $|\uparrow_z\rangle$!

We are accustomed to saying that $\boldsymbol{\rho}$ describes two (or more) different alternative pure state preparations, only one of which is actually realized each time we prepare a qubit. But we have found that what happens if we prepare $|\uparrow_z\rangle$ actually *depends on what we would have done* if we had prepared $|\downarrow_x\rangle$ instead. It is no longer sensible, apparently, to regard the two possible preparations as mutually exclusive alternatives. Evolution of the alternatives actually depends on the other alternatives that supposedly were not realized.

Joe Polchinski has called this phenomenon the "Everett phone," because the different "branches of the wave function" seem to be able to "communicate" with one another.

Nonlinear evolution of the density matrix, then, can have strange, perhaps even absurd, consequences. Even so, the argument that nonlinear evolution should be excluded is not completely compelling. Indeed Jim Hartle has argued that there are versions of "generalized quantum mechanics" in which nonlinear evolution is permitted, yet a consistent probability interpretation can be salvaged. Nevertheless, we will follow tradition here and demand that \$ be linear.

3.2.3 Complete positivity

It would be satisfying were we able to conclude that any \$ satisfying (0) - (3) has an operator-sum representation, and so can be realized by unitary evolution of a suitable bipartite system. Sadly, this is not quite possible. Happily, though, it turns out that by adding one more rather innocuous sounding assumption, we *can* show that \$ has an operator-sum representation.

The additional assumption we will need (really a stronger version of (3)) is

(3') *\$ is completely positive.*

Complete positivity is defined as follows. Consider any possible extension of $\mathcal{H}_A$ to the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$; then $\$_A$ is completely positive on $\mathcal{H}_A$ if $\$_A \otimes I_B$ is positive for all such extensions.

Complete positivity is surely a reasonable property to demand on physical grounds. If we are studying the evolution of system A, we can never be certain that there is no system B, totally uncoupled to A, of which we are unaware. Complete positivity (combined with our other assumptions) is merely the statement that, if system A evolves and system B does not, any initial density matrix of the combined system evolves to another density matrix.

We will prove that assumptions (0), (1), (2), (3′) are sufficient to ensure that \$ is a superoperator (has an operator-sum representation). (Indeed, properties (0) - (3′) can be taken as an alternative definition of a superoperator.) Before proceeding with the proof, though, we will attempt to clarify the concept of complete positivity by giving an example of a positive operator that is not completely positive. The example is the transposition operator

$$T : \boldsymbol{\rho} \to \boldsymbol{\rho}^T. \tag{3.79}$$

T preserves the eigenvalues of $\boldsymbol{\rho}$ and so clearly is positive.

But is T *completely* positive (is $T_A \otimes I_B$ necessarily positive)? Let us choose $\dim(\mathcal{H}_B) = \dim(\mathcal{H}_A) = N$, and consider the maximally entangled state

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle_A \otimes |i'\rangle_B, \tag{3.80}$$

where $\{|i\rangle_A\}$ and $\{|i'\rangle_B\}$ are orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Then

$$\begin{aligned} T_A \otimes I_B : \boldsymbol{\rho} = |\Phi\rangle_{AB\ AB}\langle\Phi| &= \frac{1}{N}\sum_{i,j}(|i\rangle_{A\ A}\langle j|) \otimes (|i'\rangle_{B\ B}\langle j'|) \\ \to \boldsymbol{\rho}' &= \frac{1}{N}\sum_{i,j}(|j\rangle_{A\ A}\langle i|) \otimes (|i'\rangle_{B\ B}\langle j'|). \end{aligned} \tag{3.81}$$

We see that the operator $N\boldsymbol{\rho}'$ acts as

$$\begin{aligned} N\boldsymbol{\rho}' :& (\sum_i a_i|i\rangle_A) \otimes (\sum_j b_j|j'\rangle_B) \\ &\to (\sum_i a_i|i'\rangle_B) \otimes (\sum_j b_j|j\rangle_A), \end{aligned} \tag{3.82}$$

or

$$N\boldsymbol{\rho}'(|\varphi\rangle_A \otimes |\psi\rangle_B) = |\psi\rangle_A \otimes |\varphi\rangle_B. \tag{3.83}$$

Hence $N\boldsymbol{\rho}'$ is a *swap operator* (which squares to the identity). The eigenstates of $N\boldsymbol{\rho}'$ are states *symmetric* under the interchange $A \leftrightarrow B$, with eigenvalue 1, and *antisymmetric* states with eigenvalue -1. Since $\boldsymbol{\rho}'$ has negative eigenvalues, it is not positive, and (since $\boldsymbol{\rho}$ is certainly positive), therefore, $T_A \otimes I_B$ does not preserve positivity. We conclude that T_A, while positive, is *not* completely positive.

3.2.4 POVM as a superoperator

A unitary transformation that entangles A with B, followed by an orthogonal measurement of B, can be described as a POVM in A. In fact, the positive operators comprising the POVM can be constructed from the Kraus operators. If $|\varphi\rangle_A$ evolves as

$$|\varphi\rangle_A|0\rangle_B \to \sum_\mu \boldsymbol{M}_\mu|\varphi\rangle_A|\mu\rangle_B, \tag{3.84}$$

then the measurement in B that projects onto the $\{|\mu\rangle_E\}$ basis has outcome μ with probability

$$\text{Prob}(\mu) = {}_A\langle\varphi|\boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu|\varphi\rangle_A. \tag{3.85}$$

Expressing $\boldsymbol{\rho}_A$ as an ensemble of pure states, we find the probability

$$\text{Prob}(\mu) = tr(\boldsymbol{F}_\mu \boldsymbol{\rho}_A), \quad \boldsymbol{F}_\mu = \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu, \tag{3.86}$$

for outcome μ; evidently $\boldsymbol{F}_\mu$ is positive, and $\sum_\mu \boldsymbol{F}_\mu = \mathbf{1}$ follows from the normalization of the Kraus operators. So this is indeed a realization of a POVM.

In particular, a POVM that modifies a density matrix according to

$$\boldsymbol{\rho} \to \sum_\mu \sqrt{\boldsymbol{F}_\mu}\boldsymbol{\rho}\sqrt{\boldsymbol{F}_\mu}, \tag{3.87}$$

is a special case of a superoperator. Since each $\sqrt{\boldsymbol{F}_\mu}$ is hermitian, the requirement

$$\sum_\mu \boldsymbol{F}_\mu = 1, \tag{3.88}$$

is just the operator-sum normalization condition. Therefore, the POVM has a "unitary representation;" there is a unitary $\mathbf{U}_{AB}$ that acts as

$$\mathbf{U}_{AB} : |\varphi\rangle_A \otimes |0\rangle_B \to \sum_\mu \sqrt{\boldsymbol{F}_\mu}|\varphi\rangle_A \otimes |\mu\rangle_B, \tag{3.89}$$

where $|\varphi\rangle_A$ is a pure state of system A. Evidently, then, by performing an orthogonal measurement in system B that projects onto the basis $\{|\mu\rangle_B\}$, we can realize the POVM that prepares

$$\boldsymbol{\rho}'_A = \frac{\sqrt{\boldsymbol{F}_\mu}\boldsymbol{\rho}_A\sqrt{\boldsymbol{F}_\mu}}{tr(\boldsymbol{F}_\mu \boldsymbol{\rho}_A)} \tag{3.90}$$

with probability

$$\text{Prob}(\mu) = tr(\boldsymbol{F}_\mu \boldsymbol{\rho}_A). \tag{3.91}$$

This implementation of the POVM is not the most efficient possible (we require a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ of dimension $N \cdot n$, if the POVM has n possible outcomes) but it is in some ways the most convenient. A POVM is the most general measurement we can perform in system A by first entangling system A with system B, and then performing an orthogonal measurement in system B.

3.3 The Kraus Representation Theorem

Now we are almost ready to prove that any \$ satisfying the conditions $(0),(1),(2)$, and $(3')$ has an operator-sum representation (the Kraus representation theorem).[5] But first we will discuss a useful trick that will be employed in the proof. It is worthwhile to describe the trick separately, because it is of wide applicability.

The trick (which we will call the "relative-state method") is to *completely* characterize an operator $\mathbf{M}_A$ acting on $\mathcal{H}_A$ by describing how $\mathbf{M}_A \otimes \mathbf{1}_B$ acts on a single pure maximally entangled state[6] in $\mathcal{H}_A \otimes \mathcal{H}_B$ (where $\dim(\mathcal{H}_B) \geq \dim(\mathcal{H}_A) \equiv N$). Consider the state

$$|\tilde{\psi}\rangle_{AB} = \sum_{i=1}^N |i\rangle_A \otimes |i'\rangle_B \tag{3.92}$$

where $\{|i\rangle_A\}$ and $\{|i'\rangle_B\}$ are orthonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_B$. (We have chosen to normalize $|\tilde{\psi}\rangle_{AB}$ so that ${}_{AB}\langle\tilde{\psi}|\tilde{\psi}\rangle_{AB} = N$; this saves us from writing various factors of $\sqrt{N}$ in the formulas below.) Note that any vector

$$|\varphi\rangle_A = \sum_i a_i|i\rangle_A, \tag{3.93}$$

in $\mathcal{H}_A$ may be expressed as a "partial" inner product

$$|\varphi\rangle_A = {}_B\langle\varphi^*|\tilde{\psi}\rangle_{AB}, \tag{3.94}$$

where

$$|\varphi^*\rangle_B = \sum_i a_i^*|i'\rangle_B. \tag{3.95}$$

We say that $|\varphi\rangle_A$ is the "relative state" of the "index state" $|\varphi^*\rangle_B$. The map

$$|\varphi\rangle_A \to |\varphi^*\rangle_B, \tag{3.96}$$

is evidently *antilinear*, and it is in fact an antiunitary map from $\mathcal{H}_A$ to a subspace of $\mathcal{H}_B$. The operator $\boldsymbol{M}_A \otimes \mathbf{1}_B$ acting on $|\tilde{\psi}\rangle_{AB}$ gives

$$(\boldsymbol{M}_A \otimes \mathbf{1}_B)|\tilde{\psi}\rangle_{AB} = \sum_i \boldsymbol{M}_A|i\rangle_A \otimes |i'\rangle_B. \tag{3.97}$$

[5] The argument given here follows B. Schumacher, quant-ph/9604023 (see Appendix A of that paper.).

[6] We say that the state $|\psi\rangle_{AB}$ is *maximally entangled* if $tr_B(|\psi\rangle_{AB}\,{}_{AB}\langle\psi|) \propto \mathbf{1}_A$.

From this state we can extract $\boldsymbol{M}_A|\psi\rangle_A$ as a relative state:

$$_B\langle\varphi^*|(\boldsymbol{M}_A \otimes \mathbf{1}_B)|\tilde{\psi}\rangle_{AB} = \boldsymbol{M}_A|\varphi\rangle_A. \tag{3.98}$$

We may interpret the relative-state formalism by saying that we can realize an ensemble of pure states in $\mathcal{H}_A$ by performing measurements in $\mathcal{H}_B$ on an entangled state – the state $|\varphi\rangle_A$ is prepared when the measurement in $\mathcal{H}_B$ has the outcome $|\varphi^*\rangle_B$. If we intend to apply an operator in $\mathcal{H}_A$, we have found that it makes no difference whether we first prepare the state and then apply the operator or we first apply the operator and then prepare the state. Of course, this conclusion makes physical sense. We could even imagine that the preparation and the operation are spacelike separated events, so that the temporal ordering has no invariant (observer-independent) meaning.

We will show that $\$_A$ has an operator-sum representation by applying the relative-state method to superoperators rather than operators. *Because* we assume that $\$_A$ is completely positive, we know that $\$_A \otimes I_B$ is positive. Therefore, if we apply $\$_A \otimes I_B$ to $\tilde{\boldsymbol{\rho}}_{AB} = |\tilde{\psi}\rangle_{AB}\ {}_{AB}\langle\tilde{\psi}|$, the result is a positive operator, an (unconventionally normalized) density matrix $\tilde{\boldsymbol{\rho}}'_{AB}$ in $\mathcal{H}_A \otimes \mathcal{H}_B$. Like any density matrix, $\tilde{\boldsymbol{\rho}}'_{AB}$ can be expanded as an ensemble of pure states. Hence we have

$$(\$_A \otimes I_B)(|\tilde{\psi}\rangle_{AB}\ {}_{AB}\langle\tilde{\psi}|) = \sum_\mu q_\mu |\tilde{\Phi}_\mu\rangle_{AB}\ {}_{AB}\langle\tilde{\Phi}_\mu|, \tag{3.99}$$

(where $q_\mu > 0$, $\sum_\mu q_\mu = 1$, and each $|\tilde{\Phi}_\mu\rangle$, like $|\tilde{\psi}\rangle_{AB}$, is normalized so that $\langle\tilde{\Phi}_\mu|\tilde{\Phi}_\mu\rangle = N$). Invoking the relative-state method, we have

$$\begin{aligned}\$_A(|\varphi\rangle_A\ {}_A\langle\varphi|) &= {}_B\langle\varphi^*|(\$_A \otimes I_B)(|\tilde{\psi}\rangle_{AB}\ {}_{AB}\langle\tilde{\psi}|)|\varphi^*\rangle_B \\ &= \sum_\mu q_\mu\ {}_B\langle\varphi^*|\tilde{\Phi}_\mu\rangle_{AB}\ {}_{AB}\langle\tilde{\Phi}_\mu|\varphi^*\rangle_B.\end{aligned} \tag{3.100}$$

Now we are almost done; we define an operator $\boldsymbol{M}_\mu$ on $\mathcal{H}_A$ by

$$\boldsymbol{M}_\mu : |\varphi\rangle_A \to \sqrt{q_\mu}\ {}_B\langle\varphi^*|\tilde{\Phi}_\mu\rangle_{AB}. \tag{3.101}$$

We can check that:

1. $\boldsymbol{M}_\mu$ is *linear*, because the map $|\varphi\rangle_A \to |\varphi^*\rangle_B$ is *antilinear*.

2. $\$_A(|\varphi\rangle_A\ {}_A\langle\varphi|) = \sum_\mu \boldsymbol{M}_\mu(|\varphi\rangle_A\ {}_A\langle\varphi|)\boldsymbol{M}_\mu^\dagger$, for any pure state $|\varphi\rangle_A \in \mathcal{H}_A$.

3. $\$_A(\boldsymbol{\rho}_A) = \sum_\mu \boldsymbol{M}_\mu \boldsymbol{\rho}_A \boldsymbol{M}_\mu^\dagger$ for any density matrix $\boldsymbol{\rho}_A$, because $\boldsymbol{\rho}_A$ can be expressed as an ensemble of pure states, and $\$_A$ is linear.

4. $\sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \mathbf{1}_A$, because $\$_A$ is trace preserving for any $\boldsymbol{\rho}_A$.

Thus, we have constructed an operator-sum representation of $\$_A$.

Put succinctly, the argument went as follows. Because $\$_A$ is completely positive, $\$_A \otimes I_B$ takes a maximally entangled density matrix on $\mathcal{H}_A \otimes \mathcal{H}_B$ to another density matrix. This density matrix can be expressed as an ensemble of pure states. With each of these pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$, we may associate (via the relative-state method) a term in the operator sum.

Viewing the operator-sum representation this way, we may quickly establish two important corollaries:

How many Kraus operators? Each $\boldsymbol{M}_\mu$ is associated with a state $|\Phi_\mu\rangle$ in the ensemble representation of $\tilde{\boldsymbol{\rho}}'_{AB}$. Since $\tilde{\boldsymbol{\rho}}'_{AB}$ has a rank at most N^2 (where $N = \dim \mathcal{H}_A$), $\$_A$ always has an operator-sum representation with at most N^2 Kraus operators.

How ambiguous? We remarked earlier that the Kraus operators

$$\mathbf{N}_a = \boldsymbol{M}_\mu U_{\mu a}, \tag{3.102}$$

(where $U_{\mu a}$ is unitary) represent the same superoperator $\$$ as the $\boldsymbol{M}_\mu$'s. Now we can see that *any* two Kraus representations of $\$$ must always be related in this way. (If there are more $\mathbf{N}_a$'s than $\boldsymbol{M}_\mu$'s, then it is understood that some zero operators are added to the $\boldsymbol{M}_\mu$'s so that the two operator sets have the same cardinality.) This property may be viewed as a consequence of the GHJW theorem.

The relative-state construction described above established a $1-1$ correspondence between ensemble representations of the (unnormalized) density matrix $(\$_A \otimes I_B)\left(|\tilde{\psi}\rangle_{AB}\ {}_{AB}\langle\tilde{\psi}|\right)$ and operator-sum representations of $\$_A$. (We explicitly described how to proceed from the ensemble representation to the operator sum, but we can clearly go the other way, too: If

$$\$_A(|i\rangle_A\ {}_A\langle j|) = \sum_\mu \boldsymbol{M}_\mu|i\rangle_A\ {}_A\langle j|\boldsymbol{M}_\mu^\dagger, \tag{3.103}$$

then

$$\begin{aligned}(\$_A \otimes I_B)(|\tilde{\psi}\rangle_{AB}\ {}_{AB}\langle\tilde{\psi}|) &= \sum_{i,j}(\boldsymbol{M}_\mu|i\rangle_A|i'\rangle_B)({}_A\langle j|\boldsymbol{M}_\mu^\dagger\ {}_B\langle j'|) \\ &= \sum_\mu q_\mu|\tilde{\Phi}_\mu\rangle_{AB}\ {}_{AB}\langle\tilde{\Phi}_\mu|,\end{aligned} \tag{3.104}$$

where

$$\sqrt{q_\mu}|\tilde{\Phi}_\mu\rangle_{AB} = \sum_i \boldsymbol{M}_\mu |i\rangle_A |i'\rangle_B .) \tag{3.105}$$

Now consider two such ensembles (or correspondingly two operator-sum representations of $\$_A$), $\{\sqrt{q_\mu}|\tilde{\Phi}_\mu\rangle_{AB}\}$ and $\{\sqrt{p_a}|\tilde{\Upsilon}_a\rangle_{AB}\}$. For each ensemble, there is a corresponding "purification" in $\mathcal{H}_{AB}\otimes\mathcal{H}_C$:

$$\sum_\mu \sqrt{q_\mu}|\tilde{\Phi}_\mu\rangle_{AB}|\alpha_\mu\rangle_C$$

$$\sum_a \sqrt{p_a}|\tilde{\Upsilon}_a\rangle_{AB}|\beta_a\rangle_C , \tag{3.106}$$

where $\{(\alpha_\mu\rangle_C\}$ and $\{|\beta_a\rangle_C\}$ are two different orthonormal sets in $\mathcal{H}_c$. The GHJW theorem asserts that these two purifications are related by $\mathbf{1}_{AB}\otimes\mathbf{U}'_C$, a unitary transformation on $\mathcal{H}_C$. Therefore,

$$\sum_a \sqrt{p_a}|\tilde{\Upsilon}_a\rangle_{AB}|\beta_a\rangle_C$$

$$\begin{aligned} &= \sum_\mu \sqrt{q_\mu}|\tilde{\Phi}_\mu\rangle_{AB}\mathbf{U}'_C|\alpha_\mu\rangle_C \\ &= \sum_{\mu,a} \sqrt{q_\mu}|\tilde{\Phi}_\mu\rangle_{AB}U_{\mu a}|\beta_a\rangle_C \ , \end{aligned} \tag{3.107}$$

where, to establish the second equality we note that the orthonormal bases $\{|\alpha_\mu\rangle_C\}$ and $\{|\beta_a\rangle_C\}$ are related by a unitary transformation, and that a product of unitary transformations is unitary. We conclude that

$$\sqrt{p_a}|\tilde{\Upsilon}_a\rangle_{AB} = \sum_\mu \sqrt{q_\mu}|\tilde{\Phi}_\mu\rangle_{AB}U_{\mu a}, \tag{3.108}$$

(where $U_{\mu a}$ is unitary) from which follows

$$\boldsymbol{N}_a = \sum_\mu \boldsymbol{M}_\mu U_{\mu a}. \tag{3.109}$$

Remark. Since we have already established that we can proceed from an operator-sum representation of \$ to a unitary representation, we have now found that any "reasonable" evolution law for density operators on $\mathcal{H}_A$ can

be realized by a unitary transformation $\mathbf{U}_{AB}$ that acts on $\mathcal{H}_A\otimes\mathcal{H}_B$ according to

$$\mathbf{U}_{AB} : |\psi\rangle_A \otimes |0\rangle_B \to \sum_\mu |\varphi\rangle_A \otimes |\mu\rangle_B . \tag{3.110}$$

Is this result surprising? Perhaps it is. We may interpret a superoperator as describing the evolution of a system (A) that interacts with its environment (B). The general states of system plus environment are entangled states. But in eq. (3.110), we have assumed an initial state of A and B that is unentangled. Apparently though a real system is bound to be entangled with its surroundings, for the purpose of describing the evolution of its density matrix there is no loss of generality if we *imagine* that there is no pre-existing entanglement when we begin to track the evolution!

Remark: The operator-sum representation provides a very convenient way to express any completely positive \$. But a positive \$ does not admit such a representation if it is not completely positive. As far as I know, there is no convenient way, comparable to the Kraus representation, to express the most general *positive* \$.

3.4 Three Quantum Channels

The best way to familiarize ourselves with the superoperator concept is to study a few examples. We will now consider three examples (all interesting and useful) of superoperators for a single qubit. In deference to the traditions and terminology of (classical) communication theory. I will refer to these superoperators as *quantum channels*. If we wish, we may imagine that \$ describes the fate of quantum information that is transmitted with some loss of fidelity from a sender to a receiver. Or, if we prefer, we may imagine (as in our previous discussion), that the transmission is in time rather than space; that is, \$ describes the evolution of a quantum system that interacts with its environment.

3.4.1 Depolarizing channel

The *depolarizing channel* is a model of a decohering qubit that has particularly nice symmetry properties. We can describe it by saying that, with probability $1-p$ the qubit remains intact, while with probability p an "error" occurs. The error can be of any one of three types, where each type of error is equally likely. If $\{|0\rangle, |1\rangle\}$ is an orthonormal basis for the qubit, the three types of errors can be characterized as:

1. Bit flip error: $\begin{smallmatrix}|0\rangle\to|1\rangle \\ |1\rangle\to|0\rangle\end{smallmatrix}$ or $|\psi\rangle \to \boldsymbol{\sigma}_1|\psi\rangle, \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

2. Phase flip error: $\begin{smallmatrix}|0\rangle\to|0\rangle \\ |1\rangle\to-|1\rangle\end{smallmatrix}$ or $|\psi\rangle \to \boldsymbol{\sigma}_3|\psi\rangle, \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

3. Both: $\begin{smallmatrix}|0\rangle\to+i|1\rangle \\ |1\rangle\to-i|0\rangle\end{smallmatrix}$ or $|\psi\rangle \to \boldsymbol{\sigma}_2|\psi\rangle, \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

If an error occurs, then $|\psi\rangle$ evolves to an ensemble of the three states $\boldsymbol{\sigma}_1|\psi\rangle, \boldsymbol{\sigma}_2|\psi\rangle, \boldsymbol{\sigma}_3|\psi\rangle$, all occuring with equal likelihood.

Unitary representation

The depolarizing channel can be represented by a unitary operator acting on $\mathcal{H}_A \otimes \mathcal{H}_E$, where $\mathcal{H}_E$ has dimension 4. (I am calling it $\mathcal{H}_E$ here to encourage you to think of the auxiliary system as the environment.) The unitary operator $\mathbf{U}_{AE}$ acts as

$$\begin{aligned}\mathbf{U}_{AE} : &|\psi\rangle_A \otimes |0\rangle_E \\ &\to \sqrt{1-p}|\psi\rangle \otimes |0\rangle_E + \sqrt{\frac{p}{3}}\Big[\boldsymbol{\sigma}_1|\psi\rangle_A \otimes |1\rangle_E \\ &+ \boldsymbol{\sigma}_2|\psi\rangle \otimes |2\rangle_E + \boldsymbol{\sigma}_3|\psi\rangle \otimes |3\rangle_E\Big].\end{aligned} \tag{3.111}$$

(Since $\mathbf{U}_{AE}$ is inner product preserving, it has a unitary extension to all of $\mathcal{H}_A \otimes \mathcal{H}_E$.) The environment evolves to one of four mutually orthogonal states that "keep a record" of what transpired; if we could only measure the environment in the basis $\{|\mu\rangle_E, \mu = 0,1,2,3\}$, we would know what kind of error had occurred (and we would be able to intervene and reverse the error).

Kraus representation

To obtain an operator-sum representation of the channel, we evaluate the partial trace over the environment in the $\{|\mu\rangle_E\}$ basis. Then

$$\boldsymbol{M}_\mu = {}_E\langle\mu|\boldsymbol{U}_{AE}|0\rangle_E, \tag{3.112}$$

so that

$$\boldsymbol{M}_0 = \sqrt{1-p}\,\mathbf{1},\ \boldsymbol{M}_, = \sqrt{\frac{p}{3}}\,\boldsymbol{\sigma}_1,\ \boldsymbol{M}_2 = \sqrt{\frac{p}{3}}\,\boldsymbol{\sigma}_2,\ \boldsymbol{M}_3 = \sqrt{\frac{p}{3}}\,\boldsymbol{\sigma}_3. \tag{3.113}$$

Using $\boldsymbol{\sigma}_i^2 = \mathbf{1}$, we can readily check the normalization condition

$$\sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \left[(1-p) + 3\frac{p}{3}\right]\mathbf{1} = \mathbf{1}. \tag{3.114}$$

A general initial density matrix $\boldsymbol{\rho}_A$ of the qubit evolves as

$$\begin{aligned}\boldsymbol{\rho} \to \boldsymbol{\rho}' = (1-p)\boldsymbol{\rho} + \\ \frac{p}{3}(\boldsymbol{\sigma}_1\boldsymbol{\rho}\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2\boldsymbol{\rho}\boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3\boldsymbol{\rho}\boldsymbol{\sigma}_3).\end{aligned} \tag{3.115}$$

where we are summing over the four (in principle distinguishable) ways that the environment could evolve.

Relative-state representation

We can also characterize the channel by describing how a maximally-entangled state of two qubits evolves, when the channel acts only on the first qubit. There are four mutually orthogonal maximally entangled states, which may be denoted

$$\begin{aligned}|\phi^+\rangle_{AB} &= \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}), \\ |\phi^-\rangle_{AB} &= \frac{1}{\sqrt{2}}(|00\rangle_{AB} - |11\rangle_{AB}), \\ |\psi^+\rangle_{AB} &= \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB}), \\ |\psi^-\rangle_{AB} &= \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB}).\end{aligned} \tag{3.116}$$

If the initial state is $|\phi^+\rangle_{AB}$, then when the depolarizing channel acts on the first qubit, the entangled state evolves as

$$|\phi^+\rangle\langle\phi^+| \to (1-p)|\phi^+\rangle\langle\phi^+|$$

$$+\frac{p}{3}\left(|\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-| + |\phi^-\rangle\langle\phi^-|\right). \tag{3.117}$$

The "worst possible" quantum channel has $p = 3/4$ for in that case the initial entangled state evolves as

$$|\phi^+\rangle\langle\phi^+| \to \frac{1}{4}\left(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-|\right.$$

$$\left.+|\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|\right) = \frac{1}{4}\mathbf{1}_{AB}; \tag{3.118}$$

it becomes the totally random density matrix on $\mathcal{H}_A \otimes \mathcal{H}_B$. By the relative-state method, then, we see that a pure state $|\varphi\rangle_A$ of qubit A evolves as

$$|\varphi\rangle_A\ {}_A\langle\varphi| \to\ {}_B\langle\varphi^*|2\left(\frac{1}{4}\mathbf{1}_{AB}\right)|\varphi^*\rangle_B = \frac{1}{2}\mathbf{1}_A; \tag{3.119}$$

it becomes the random density matrix on $\mathcal{H}_A$, irrespective of the value of the initial state $|\varphi\rangle_A$. It is as though the channel threw away the initial quantum state, and replaced it by completely random junk.

An alternative way to express the evolution of the maximally entangled state is

$$|\phi^+\rangle\langle\phi^+| \to \left(1-\frac{4}{3}p\right)|\phi^+\rangle\langle\phi^+| + \frac{4}{3}p\left(\frac{1}{4}\mathbf{1}_{AB}\right). \tag{3.120}$$

Thus instead of saying that an error occurs with probability p, with errors of three types all equally likely, we could instead say that an error occurs with probability $4/3p$, where the error completely "randomizes" the state (at least we can say that for $p \leq 3/4$). The existence of two natural ways to define an "error probability" for this channel can sometimes cause confusion and misunderstanding.

One useful measure of how well the channel preserves the original quantum information is called the "entanglement fidelity" F_e. It quantifies how "close" the final density matrix is to the original maximally entangled state $|\phi^+\rangle$:

$$F_e = \langle\phi^+|\boldsymbol{\rho}'|\phi^+\rangle. \tag{3.121}$$

For the depolarizing channel, we have $F_e = 1 - p$, and we can interpret F_e as the probability that no error occured.

Block-sphere representation

It is also instructive to see how the depolarizing channel acts on the Bloch sphere. An arbitrary density matrix for a single qubit can be written as

$$\boldsymbol{\rho} = \frac{1}{2}\left(\mathbf{1} + \vec{P}\cdot\vec{\boldsymbol{\sigma}}\right), \tag{3.122}$$

where $\vec{P}$ is the "spin polarization" of the qubit. Suppose we rotate our axes so that $\vec{P} = P_3\hat{e}_3$ and $\boldsymbol{\rho} = \frac{1}{2}(\mathbf{1} + P_3\boldsymbol{\sigma}_3)$. Then, since $\boldsymbol{\sigma}_3\boldsymbol{\sigma}_3\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 = -\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3\boldsymbol{\sigma}_2$, we find

$$\boldsymbol{\rho}' = \left(1 - p + \frac{p}{3}\right)\frac{1}{2}(\mathbf{1} + P_3\boldsymbol{\sigma}_3) + \frac{2p}{3}\frac{1}{2}(\mathbf{1} - P_3\boldsymbol{\sigma}_3), \tag{3.123}$$

or $P_3' = \left(1 - \frac{4}{3}p\right)P_3$. From the rotational symmetry, we see that

$$\vec{P}' = \left(1 - \frac{4}{3}p\right)\vec{P}, \tag{3.124}$$

irrespective of the direction in which P points. Hence, the Bloch sphere contracts uniformly under the action of the channel; the spin polarization is reduced by the factor $1 - \frac{4}{3}p$ (which is why we call it the depolarizing channel). This result was to be expected in view of the observation above that the spin is totally "randomized" with probability $\frac{4}{3}p$.

Invertibility?

Why do we say that the superoperator is not invertible? Evidently we can reverse a uniform contraction of the sphere with a uniform inflation. But the trouble is that the inflation of the Bloch sphere is not a superoperator, because it is not positive. Inflation will take values of $\vec{P}$ with $|\vec{P}| \leq 1$ to values with $|\vec{P}| > 1$, and so will take a density operator to an operator with a negative eigenvalue. Decoherence can shrink the ball, but no physical process can blow it up again! A superoperator running backwards in time is *not* a superoperator.

3.4.2 Phase-damping channel

Our next example is the *phase-damping channel*. This case is particularly instructive, because it provides a revealing caricature of decoherence in realistic physical situations, with all inessential mathematical details stripped away.

Unitary representation

A unitary representation of the channel is

$$\begin{aligned} |0\rangle_A|0\rangle_E &\to \sqrt{1-p}|0\rangle_A|0\rangle_E + \sqrt{p}|0\rangle_A|1\rangle_E, \\ |1\rangle_A|0\rangle_E &\to \sqrt{1-p}|1\rangle_A|0\rangle_E + \sqrt{p}|1\rangle_A|2\rangle_E. \end{aligned} \tag{3.125}$$

In this case, unlike the depolarizing channel, qubit A does not make any transitions. Instead, the environment "scatters" off of the qubit occasionally (with probability p) being kicked into the state $|1\rangle_E$ if A is in the state $|0\rangle_A$ and into the state $|2\rangle_E$ if A is in the state $|1\rangle_A$. Furthermore, also unlike the depolarizing channel, the channel picks out a preferred basis for qubit A; the basis $\{|0\rangle_A, |1\rangle_A\}$ is the only basis in which bit flips never occur.

Kraus operators

Evaluating the partial trace over $\mathcal{H}_E$ in the $\{|0\rangle_E, |1\rangle_E, |2\rangle_E\}$basis, we obtain the Kraus operators

$$\boldsymbol{M}_0 = \sqrt{1-p}\mathbf{1}, \boldsymbol{M}_1 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \boldsymbol{M}_2 = \sqrt{p}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.126}$$

it is easy to check that $\boldsymbol{M}_0^2 + \boldsymbol{M}_1^2 + \boldsymbol{M}_2^2 = \mathbf{1}$. In this case, three Kraus operators are not really needed; a representation with two Kraus operators is possible, as you will show in a homework exercise.

An initial density matrix $\boldsymbol{\rho}$ evolves to

$$\begin{aligned} \$(\boldsymbol{\rho}) &= \boldsymbol{M}_0\boldsymbol{\rho}\boldsymbol{M}_0 + \boldsymbol{M}_1\boldsymbol{\rho}\boldsymbol{M}_1 + \boldsymbol{M}_2\boldsymbol{\rho}\boldsymbol{M}_2 \\ &= (1-p)\boldsymbol{\rho} + p\begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & (1-p) \quad \rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}; \end{aligned} \tag{3.127}$$

thus the on-diagonal terms in $\boldsymbol{\rho}$ remain invariant while the off-diagonal terms decay.

Now suppose that the probability of a scattering event per unit time is Γ, so that $p = \Gamma\Delta t \ll 1$ when time Δt elapses. The evolution over a time

$t = n\Delta t$ is governed by $\$^n$, so that the off-diagonal terms are suppressed by $(1-p)^n = (1-\Gamma\Delta t)^{t/\Delta t} \to e^{-\Gamma t}$ (as $\Delta t \to 0$). Thus, if we prepare an initial pure state $a|0\rangle + b|1\rangle$, then after a time $t \gg \Gamma^{-1}$, the state decays to the incoherent superposition $\boldsymbol{\rho}' = |a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1|$. Decoherence occurs, in the preferred basis $\{|0\rangle, |1\rangle\}$.

Bloch-sphere representation

This will be worked out in a homework exercise.

Interpretation

We might interpret the phase-damping channel as describing a heavy "classical" particle (e.g., an interstellar dust grain) interacting with a background gas of light particles (e.g., the 3^0K microwave photons). We can imagine that the dust is initially prepared in a superposition of position eigenstates $|\psi\rangle = \frac{1}{\sqrt{2}}(|x\rangle + |-x\rangle)$ (or more generally a superposition of position-space wavepackets with little overlap). We might be able to monitor the behavior of the dust particle, but it is hopeless to keep track of the quantum state of all the photons that scatter from the particle; for our purposes, the quantum state of the particle is described by the density matrix $\boldsymbol{\rho}$ obtained by tracing over the photon degrees of freedom.

Our analysis of the phase damping channel indicates that if photons are scattered by the dust particle at a rate Γ, then the off-diagonal terms in $\boldsymbol{\rho}$ decay like $\exp(-\Gamma t)$, and so become completely negligible for $t \gg \Gamma^{-1}$. At that point, the coherence of the superposition of position eigenstates is completely lost – there is no chance that we can recombine the wavepackets and induce them to interfere. (If we attempt to do a double-slit interference pattern with dust grains, we will not see any interference pattern if it takes a time $t \gg \Gamma^{-1}$ for the grain to travel from the source to the screen.)

The dust grain is heavy. Because of its large inertia, its state of motion is little affected by the scattered photons. Thus, there are two disparate time scales relevant to its dynamics. On the one hand, there is a damping time scale, the time for a significant amount of the particle's momentum to be transfered to the photons; this is a long time if the particle is heavy. On the other hand, there is the decoherence time scale. In this model, the time scale for decoherence is of order Γ, the time for a *single* photon to be scattered by the dust grain, which is far shorter than the damping time scale. For a macroscopic object, decoherence is *fast*.

As we have already noted, the phase-damping channel picks out a preferred basis for decoherence, which in our "interpretation" we have assumed to be the position-eigenstate basis. Physically, decoherence prefers the spatially localized states of the dust grain because the *interactions* of photons and grains are localized in space. Grains in distinguishable positions tend to scatter the photons of the environment into mutually orthogonal states.

Even if the separation between the "grains" were so small that it could not be resolved very well by the scattered photons, the decoherence process would still work in a similar way. Perhaps photons that scatter off grains at positions x and $-x$ are not mutually orthogonal, but instead have an overlap

$$\langle \gamma + | \gamma - \rangle = 1 - \varepsilon, \varepsilon \ll 1. \tag{3.128}$$

The phase-damping channel would still describe this situation, but with p replaced by $p\varepsilon$ (if p is still the probability of a scattering event). Thus, the decoherence rate would become $\Gamma_{\mathbf{dec}} = \varepsilon \Gamma_{\mathbf{scat}}$, where $\Gamma_{\mathbf{scat}}$ is the scattering rate (see the homework).

The intuition we distill from this simple model applies to a vast variety of physical situations. A coherent superposition of macroscopically distinguishable states of a "heavy" object decoheres very rapidly compared to its damping rate. The spatial locality of the interactions of the system with its environment gives rise to a preferred "local" basis for decoherence. Presumably, the same principles would apply to the decoherence of a "cat state" $\frac{1}{\sqrt{2}}(|\text{ dead}\rangle + |\text{ alive}\rangle)$, since "deadness" and "aliveness" can be distinguished by localized probes.

3.4.3 Amplitude-damping channel

The *amplitude-damping channel* is a schematic model of the decay of an excited state of a (two-level) atom due to spontaneous emission of a photon. By detecting the emitted photon ("observing the environment") we can perform a POVM that gives us information about the initial preparation of the atom.

Unitary representation

We denote the atomic ground state by $|0\rangle_A$ and the excited state of interest by $|1\rangle_A$. The "environment" is the electromagnetic field, assumed initially to be in its vacuum state $|0\rangle_E$. After we wait a while, there is a probability p

that the excited state has decayed to the ground state and a photon has been emitted, so that the environment has made a transition from the state $|0\rangle_E$ ("no photon") to the state $|1\rangle_E$ ("one photon"). This evolution is described by a unitary transformation that acts on atom and environment according to

$$\begin{aligned} &|0\rangle_A|0\rangle_E \to |0\rangle_A|0\rangle_E \\ &|1\rangle_A|0\rangle_E \to \sqrt{1-p}|1\rangle_A|0\rangle_E + \sqrt{p}|0\rangle_A|1\rangle_E. \end{aligned} \tag{3.129}$$

(Of course, if the atom starts out in its ground state, and the environment is at zero temperature, then there is no transition.)

Kraus operators

By evaluating the partial trace over the environment in the basis $\{|0\rangle_E, |1\rangle_E\}$, we find the kraus operators

$$\boldsymbol{M}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \boldsymbol{M}_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}, \tag{3.130}$$

and we can check that

$$\boldsymbol{M}_0^\dagger \boldsymbol{M}_0 + \boldsymbol{M}_1^\dagger \boldsymbol{M}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1-p \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} = \mathbf{1}. \tag{3.131}$$

The operator $\boldsymbol{M}_1$ induces a "quantum jump" – the decay from $|1\rangle_A$ to $|0\rangle_A$, and $\boldsymbol{M}_0$ describes how the state evolves if no jump occurs. The density matrix evolves as

$$\begin{aligned} \boldsymbol{\rho} \to \$(\rho) &= \boldsymbol{M}_0 \boldsymbol{\rho} \boldsymbol{M}_0^\dagger + \boldsymbol{M}_1 \boldsymbol{\rho} \boldsymbol{M}_1^\dagger \\ &= \begin{pmatrix} \rho_{00} & \sqrt{1-p}\rho_{01} \\ \sqrt{1-p}\rho_{10} & (1-p)\rho_{11} \end{pmatrix} + \begin{pmatrix} p\rho_{11} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\rho_{01} \\ \sqrt{1-p}\rho_{10} & (1-p)\rho_{11} \end{pmatrix}. \end{aligned} \tag{3.132}$$

If we apply the channel n times in succession, the ρ_{11} matrix element decays as

$$\rho_{11} \to (1-p)^n \rho_{11}; \tag{3.133}$$

so if the probability of a transition in time interval Δt is $\Gamma\Delta t$, then the probability that the excited state persists for time t is $(1-\Gamma\Delta t)^{t/\Delta t} \to e^{-\Gamma t}$, the expected exponential decay law.

As $t \to \infty$, the decay probability approaches unity, so

$$\$(\boldsymbol{\rho}) \to \begin{pmatrix} \rho_{00}+\rho_{11} & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.134}$$

The atom always winds up in its ground state. This example shows that it is sometimes possible for a superoperator to take a mixed initial state, e.g.,

$$\boldsymbol{\rho} = \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix}, \tag{3.135}$$

to a pure final state.

Watching the environment

In the case of the decay of an excited atomic state via photon emission, it may not be impractical to monitor the environment with a photon detector. The measurement of the environment prepares a pure state of the atom, and so in effect prevents the atom from decohering.

Returning to the unitary representation of the amplitude-damping channel, we see that a coherent superposition of the atomic ground and excited states evolves as

$$\begin{aligned}&(a|0\rangle_A + b|1\rangle_A)|0\rangle_E \\ &\quad\to (a|0\rangle_A + b\sqrt{1-p}|1\rangle)|0\rangle_E + \sqrt{p}|0\rangle_A|1\rangle_E.\end{aligned} \tag{3.136}$$

If we detect the photon (and so project out the state $|1\rangle_E$ of the environment), then we have prepared the state $|0\rangle_A$ of the atom. In fact, we have prepared a state in which we know with certainty that the initial atomic state was the excited state $|1\rangle_A$ – the ground state could not have decayed.

On the other hand, if we detect no photon, and our photon detector has perfect efficiency, then we have projected out the state $|0\rangle_E$ of the environment, and so have prepared the atomic state

$$a|0\rangle_A + b\sqrt{1-p}|1\rangle_A. \tag{3.137}$$

The atomic state has evolved due to our failure to detect a photon – it has become more likely that the initial atomic state was the ground state!

As noted previously, a unitary transformation that entangles A with E, followed by an orthogonal measurement of E, can be described as a POVM in A. If $|\varphi\rangle_A$ evolves as

$$|\varphi\rangle_A|0\rangle_E \to \sum_\mu \boldsymbol{M}_\mu|\varphi\rangle_A|\mu\rangle_E, \tag{3.138}$$

then an orthogonal measurement in E that projects onto the $\{|\mu\rangle_E\}$ basis realizes a POVM with

$$\mathrm{Prob}(\mu) = tr(\boldsymbol{F}_\mu\boldsymbol{\rho}_A), \quad \boldsymbol{F}_\mu = \boldsymbol{M}_\mu^\dagger\boldsymbol{M}_\mu, \tag{3.139}$$

for outcome μ. In the case of the amplitude damping channel, we find

$$\boldsymbol{F}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1-p \end{pmatrix}, \quad \boldsymbol{F}_1 = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}, \tag{3.140}$$

where $\boldsymbol{F}_1$ determines the probability of a successful photon detection, and $\boldsymbol{F}_0$ the complementary probability that no photon is detected.

If we wait a time $t \gg \Gamma^{-1}$, so that p approaches 1, our POVM approaches an orthogonal measurement, the measurement of the initial atomic state in the $\{|0\rangle_A, |1\rangle_A\}$ basis. A peculiar feature of this measurement is that we can project out the state $|0\rangle_A$ by *not* detecting a photon. This is an example of what Dicke called "interaction-free measurement" – because *no change* occured in the state of the environment, we can infer what the atomic state must have been. The term "interaction-free measurement" is in common use, but it is rather misleading; obviously, if the Hamiltonian of the world did not include a coupling of the atom to the electromagnetic field, the measurement could not have been possible.

3.5 Master Equation

3.5.1 Markovian evolution

The superoperator formalism provides us with a general description of the evolution of density matrices, including the evolution of pure states to mixed states (decoherence). In the same sense, unitary transformations provide a general description of coherent quantum evolution. But in the case of coherent evolution, we find it very convenient to characterize the dynamics of a quantum system with a *Hamiltonian*, which describes the evolution over an infinitesimal time interval. The dynamics is then described by a differential equation, the *Schrödinger equation*, and we may calculate the evolution over a finite time interval by integrating the equation, that is, by piecing together the evolution over many infinitesimal intervals.

It is often possible to describe the (not necessarily coherent) evolution of a density matrix, at least to a good approximation, by a differential equation. This equation, the *master equation*, will be our next topic.

In fact, it is not at all obvious that there need be a differential equation that describes decoherence. Such a description will be possible only if the evolution of the quantum system is "Markovian," or in other words, *local* in time. If the evolution of the density operator $\boldsymbol{\rho}(t)$ is governed by a (first-order) differential equation in t, then that means that $\boldsymbol{\rho}(t+dt)$ is completely determined by $\boldsymbol{\rho}(t)$.

We have seen that we can always describe the evolution of density operator $\boldsymbol{\rho}_A$ in Hilbert space $\mathcal{H}_A$ if we imagine that the evolution is actually unitary in the extended Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_E$. But even if the evolution in $\mathcal{H}_A \otimes \mathcal{H}_E$ is governed by a Schrdinger equation, this is not sufficient to ensure that the evolution of $\boldsymbol{\rho}_A(t)$ will be local in t. Indeed, if we know only $\boldsymbol{\rho}_A(t)$, we do not have complete initial data for the Schrodinger equation; we need to know the state of the "environment," too. Since we know from the general theory of superoperators that we are entitled to insist that the quantum state in $\mathcal{H}_A \otimes \mathcal{H}_E$ at time $t = 0$ is

$$\boldsymbol{\rho}_A \otimes |0\rangle_E \, {}_E\langle 0|, \tag{3.141}$$

a sharper statement of the difficulty is that the density operator $\boldsymbol{\rho}_A(t+dt)$ depends not only on $\boldsymbol{\rho}_A(t)$, but also on $\boldsymbol{\rho}_A$ at earlier times, because the reservoir E[7] retains a *memory* of this information for a while, and can transfer it back to system A.

This quandary arises because information flows on a two-way street. An open system (whether classical or quantum) is *dissipative* because information can flow from the system to the reservoir. But that means that information can also flow back from reservoir to system, resulting in non-Markovian

[7] In discussions of the mater equation, the environment is typically called the *reservoir*, in deference to the deeply ingrained conventions of statistical physics.

fluctuations of the system.[8]

Except in the case of coherent (unitary) evolution, then, fluctuations are inevitable, and an exact Markovian description of quantum dynamics is impossible. Still, in many contexts, a Markovian description is a very good approximation. The key idea is that there may be a clean separation between the typical correlation time of the fluctuations and the time scale of the evolution that we want to follow. Crudely speaking, we may denote by $(\Delta t)_{\text{res}}$ the time it takes for the reservoir to "forget" information that it acquired from the system — after time $(\Delta t)_{\text{res}}$ we can regard that information as forever lost, and neglect the possibility that the information may feed back again to influence the subsequent evolution of the system.

Our description of the evolution of the system will incorporate "coarse-graining" in time; we perceive the dynamics through a filter that screens out the high frequency components of the motion, with $\omega \gg (\Delta t_{\text{coarse}})^{-1}$. An approximately Markovian description should be possible, then, if $(\Delta t)_{\text{res}} \ll (\Delta t)_{\text{coarse}}$; we can neglect the memory of the reservoir, because we are unable to resolve its effects. This "Markovian approximation" will be *useful* if the time scale of the dynamics that we want to observe is long compared to $(\Delta t)_{\text{coarse}}$, e.g., if the *damping* time scale $(\Delta t)_{\text{damp}}$ satisfies

$$(\Delta t)_{\text{damp}} \gg (\Delta t)_{\text{coarse}} \gg (\Delta t)_{\text{res}}. \tag{3.142}$$

This condition often applies in practice, for example in atomic physics, where $(\Delta t)_{\text{res}} \sim \hbar/kT \sim 10^{-14}$ s (T is the temperature) is orders of magnitude larger than the typical lifetime of an excited atomic state.

An instructive example to study is the case where the system A is a single harmonic oscillator ($\mathbf{H}_A = \omega \boldsymbol{a}^\dagger \boldsymbol{a}$), and the reservoir R consists of many oscillators ($\boldsymbol{H}_R = \sum_i \omega_i \boldsymbol{b}_i^\dagger \boldsymbol{b}_i$, weakly coupled to the system by a perturbation

$$\mathbf{H}' = \sum_i \lambda_i (\boldsymbol{a}\boldsymbol{b}_i^\dagger + \boldsymbol{a}^\dagger \boldsymbol{b}_i). \tag{3.143}$$

The reservoir Hamiltonian could represent the (free) electromagnetic field, and then $\boldsymbol{H}'$, in lowest nontrivial order of perturbation theory induces transitions in which the oscillator emits or absorbs a single photon, with its occupation number $n = \boldsymbol{a}^\dagger \boldsymbol{a}$ decreasing or increasing accordingly.

[8] This inescapable connection underlies the *fluctuation-dissipation theorem*, a powerful tool in statistical physics.

We could arrive at the master equation by analyzing this system using time-dependent perturbation theory, and carefully introducing a finite frequency cutoff. The details of that analysis can be found in the book "An Open Systems Approach to Quantum Optics," by Howard Carmichael. Here, though, I would like to short-circuit that careful analysis, and leap to the master equation by a more heuristic route.

3.5.2 The Lindbladian

Under unitary evolution, the time evolution of the density matrix is governed by the Schrödinger equation

$$\dot{\boldsymbol{\rho}} = -i[\boldsymbol{H}, \boldsymbol{\rho}], \tag{3.144}$$

which we can solve formally to find

$$\boldsymbol{\rho}(t) = e^{-i\boldsymbol{H}t}\boldsymbol{\rho}_{(0)}e^{i\boldsymbol{H}t}, \tag{3.145}$$

if $\boldsymbol{H}$ is time independent. Our goal is to generalize this equation to the case of Markovian but nonunitary evolution, for which we will have

$$\dot{\boldsymbol{\rho}} = \mathcal{L}[\boldsymbol{\rho}]. \tag{3.146}$$

The linear operator $\mathcal{L}$, which generates a finite superoperator in the same sense that a Hamiltonian $\boldsymbol{H}$ generates unitary time evolution, will be called the *Lindbladian*. The formal solution to eq. (3.146) is

$$\boldsymbol{\rho}(t) = e^{\mathcal{L}t}[\boldsymbol{\rho}(0)], \tag{3.147}$$

if $\mathcal{L}$ is t-independent.

To compute the Lindbladian, we could start with the Schrödinger equation for the coupled system and reservoir

$$\dot{\boldsymbol{\rho}}_A = tr_R(\dot{\boldsymbol{\rho}}_{AR}) = tr_R(-i[\boldsymbol{H}_{AR}, \boldsymbol{\rho}_{AR}]), \tag{3.148}$$

but as we have already noted, we cannot expect that this formula for $\dot{\boldsymbol{\rho}}_A$ can be expressed in terms of $\boldsymbol{\rho}_A$ alone. To obtain the Lindbladian, we need to explicitly invoke the Markovian approximation (as Carmichael does). On the other hand, suppose we *assume* that the Markov approximation applies. We already know that a *general* superoperator has a Kraus representation

$$\boldsymbol{\rho}(t) = \$_t(\boldsymbol{\rho}(0)) = \sum_\mu \boldsymbol{M}_\mu(t)\boldsymbol{\rho}(0)\boldsymbol{M}_\mu^\dagger(t), \tag{3.149}$$

and that $\$_{t=0} = I$. If the elapsed time is the infinitesimal interval dt, and

$$\boldsymbol{\rho}(dt) = \boldsymbol{\rho}(0) + O(dt), \tag{3.150}$$

then one of the Kraus operators will be $\boldsymbol{M}_0 = \mathbf{1} + O(dt)$, and all the others will be of order $\sqrt{dt}$. The operators $\boldsymbol{M}_\mu, \mu > 0$ describe the "quantum jumps" that the system might undergo, all occuring with a probability of order dt. We may, therefore, write

$$\begin{aligned} \boldsymbol{M}_\mu &= \sqrt{dt}\,\mathbf{L}_\mu, \quad \mu = 1, 2, 3, \ldots \\ \boldsymbol{M}_0 &= \mathbf{1} + (-i\boldsymbol{H} + \mathbf{K})dt, \end{aligned} \tag{3.151}$$

where $\boldsymbol{H}$ and $\boldsymbol{K}$ are both hermitian and $\boldsymbol{L}_\mu, \boldsymbol{H}$, and $\boldsymbol{K}$ are all zeroth order in dt. In fact, we can determine $\mathbf{K}$ by invoking the Kraus normalization condition:

$$\mathbf{1} = \sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \mathbf{1} + dt(2\boldsymbol{K} + \sum_{\mu>0} \boldsymbol{L}_\mu^\dagger \boldsymbol{L}_\mu), \tag{3.152}$$

or

$$\boldsymbol{K} = -\frac{1}{2}\sum_{\mu>0} \boldsymbol{L}_\mu^\dagger \boldsymbol{L}_\mu. \tag{3.153}$$

Substituting into eq. (3.149), expressing $\boldsymbol{\rho}(dt) = \boldsymbol{\rho}(0) + dt\dot{\boldsymbol{\rho}}(0)$, and equating terms of order dt, we obtain Lindblad's equation:

$$\dot{\boldsymbol{\rho}} \equiv \mathcal{L}[\boldsymbol{\rho}] = -i[\boldsymbol{H}, \boldsymbol{\rho}] + \sum_{\mu>0}\left(\boldsymbol{L}_\mu\boldsymbol{\rho}\boldsymbol{L}_\mu^\dagger - \frac{1}{2}\boldsymbol{L}_\mu^\dagger\boldsymbol{L}_\mu\boldsymbol{\rho} - \frac{1}{2}\boldsymbol{\rho}\boldsymbol{L}_\mu^\dagger\boldsymbol{L}_\mu\right). \tag{3.154}$$

The first term in $\mathcal{L}[\boldsymbol{\rho}]$ is the usual Schrodinger term that generates unitary evolution. The other terms describe the possible transitions that the system may undergo due to interactions with the reservoir. The operators $\boldsymbol{L}_\mu$ are called *Lindblad operators* or *quantum jump operators*. Each $\boldsymbol{L}_\mu\boldsymbol{\rho}\boldsymbol{L}_\mu^\dagger$ term induces one of the possible quantum jumps, while the $-1/2\boldsymbol{L}_\mu^\dagger\boldsymbol{L}_\mu\boldsymbol{\rho} - 1/2\boldsymbol{\rho}\boldsymbol{L}_\mu^\dagger\boldsymbol{L}_\mu$ terms are needed to normalize properly the case in which no jumps occur.

Lindblad's eq (3.154) is what we were seeking – the general form of (completely positive) Markovian evolution of a density matrix: that is, the master equation. It follows from the Kraus representation that we started with that Lindblad's equation preserves density matrices: $\boldsymbol{\rho}(t + dt)$ is a density matrix if $\boldsymbol{\rho}(t)$ is. Indeed, we can readily check, using eq. (3.154), that $\dot{\boldsymbol{\rho}}$ is Hermitian and $tr\dot{\boldsymbol{\rho}} = 0$. That $\mathcal{L}[\boldsymbol{\rho}]$ preserves positivity is somewhat less manifest but, as already noted, follows from the Kraus representation.

If we recall the connection between the Kraus representation and the unitary representation of a superoperator, we clarify the interpretation of the master equation. We may imagine that we are continuously monitoring the reservoir, projecting it in each instant of time onto the $|\mu\rangle_R$ basis. With probability $1 - 0(dt)$, the reservoir remains in the state $|0\rangle_R$, but with probability of order dt, the reservoir makes a quantum jump to one of the states $|\mu\rangle_R$, $\mu > 0$. When we say that the reservoir has "forgotten" the information it acquired from the system (so that the Markovian approximation applies), we mean that these transitions occur with probabilities that increase linearly with time. Recall that this is *not* automatic in time-dependent perturbation theory. At a small time t the probability of a particular transition is proportional to t^2; we obtain a rate (in the derivation of "Fermi's golden rule") only by summing over a continuum of possible final states. Because the number of accessible states actually decreases like $1/t$, the probability of a transition, summed over final states, is proportional to t. By using a Markovian description of dynamics, we have implicitly assumed that our $(\Delta t)_{\rm coarse}$ is long enough so that we can assign rates to the various possible transitions that might be detected when we monitor the environment. In practice, this is where the requirement $(\Delta t)_{\rm coarse} \gg (\Delta t)_{\rm res}$ comes from.

3.5.3 Damped harmonic oscillator

As an example to illustrate the master equation, we consider the case of a harmonic oscillator interacting with the electromagnetic field, coupled as

$$\boldsymbol{H}' = \sum_i \lambda_i (\boldsymbol{a}\boldsymbol{b}_i^\dagger + \boldsymbol{a}^\dagger \boldsymbol{b}_i). \tag{3.155}$$

Let us also suppose that the reservoir is at zero temperature; then the excitation level of the oscillator can cascade down by successive emission of photons, but no absorption of photons will occur. Hence, there is only one jump operator:

$$\boldsymbol{L}_1 = \sqrt{\Gamma}\boldsymbol{a}. \tag{3.156}$$

Here Γ is the rate for the oscillator to decay from the first excited ($n = 1$) state to the ground ($n = 0$) state; because of the form of $\boldsymbol{H}$, the rate for

the decay from level n to $n - I$ is $n\Gamma$.[9] The master equation in the Lindblad form becomes

$$\dot{\boldsymbol{\rho}} = -i[\boldsymbol{H}_0, \boldsymbol{\rho}] + \Gamma(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \frac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}). \tag{3.157}$$

where $\boldsymbol{H}_0 = \omega\boldsymbol{a}^\dagger\boldsymbol{a}$ is the Hamiltonian of the oscillator. This is the same equation obtained by Carmichael from a more elaborate analysis. (The only thing we have missed is the *Lamb shift*, a radiative renormalization of the frequency of the oscillator that is of the same order as the jump terms in $\mathcal{L}[\boldsymbol{\rho}]$.)

The jump terms in the master equation describe the *damping* of the oscillator due to photon emission.[10] To study the effect of the jumps, it is convenient to adopt the *interaction picture*; we define interaction picture operators $\boldsymbol{\rho}_I$ and $\boldsymbol{a}_I$ by

$$\begin{aligned} \boldsymbol{\rho}(t) &= e^{-i\boldsymbol{H}_0 t}\boldsymbol{\rho}_I(t)e^{i\boldsymbol{H}_0 t}, \\ \boldsymbol{a}(t) &= e^{-i\boldsymbol{H}_0 t}\boldsymbol{a}_I(t)e^{i\boldsymbol{H}_0 t}, \end{aligned} \tag{3.158}$$

so that

$$\dot{\boldsymbol{\rho}}_I = \Gamma(\boldsymbol{a}_I\boldsymbol{\rho}_I\boldsymbol{a}_I^\dagger - \frac{1}{2}\boldsymbol{a}_I^\dagger\boldsymbol{a}_I\boldsymbol{\rho} - \frac{1}{2}\boldsymbol{\rho}_I\boldsymbol{a}_I^\dagger\boldsymbol{a}_I). \tag{3.159}$$

where in fact $\boldsymbol{a}_I(t) = \boldsymbol{a}e^{-i\omega t}$ so we can replace $\boldsymbol{a}_I$ by $\boldsymbol{a}$ on the right-hand side. The variable $\tilde{\boldsymbol{a}} = e^{-iH_0 t}\boldsymbol{a}e^{+iH_0 t} = e^{i\omega t}\boldsymbol{a}$ remains constant in the absence of damping. With damping, $\tilde{\boldsymbol{a}}$ decays according to

$$\frac{d}{dt}\langle\tilde{\boldsymbol{a}}\rangle = \frac{d}{dt}tr(\boldsymbol{a}\boldsymbol{\rho}_I) = tr\boldsymbol{a}\dot{\boldsymbol{\rho}}\ , \tag{3.160}$$

and from eq. (3.159) we have

$$tr\boldsymbol{a}\dot{\boldsymbol{\rho}} = \Gamma tr\left(\boldsymbol{a}^2\boldsymbol{\rho}_I\boldsymbol{a}^\dagger - \frac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho}_I - \frac{1}{2}\boldsymbol{a}\boldsymbol{\rho}_I\boldsymbol{a}^\dagger\boldsymbol{a}\right)$$

[9] The nth level of excitation of the oscillator may be interpreted as a state of n noninteracting particles; the rate is $n\Gamma$ because any one of the n particles can decay.

[10] This model extends our discussion of the amplitude-damping channel to a damped oscillator rather than a damped qubit.

$$= \Gamma tr\left(\frac{1}{2}[\boldsymbol{a}^\dagger,\boldsymbol{a}]\boldsymbol{a}\boldsymbol{\rho}_I\right) = -\frac{\Gamma}{2}tr(\boldsymbol{a}\boldsymbol{\rho}_I) = -\frac{\Gamma}{2}\langle\tilde{\boldsymbol{a}}\rangle. \tag{3.161}$$

Integrating this equation, we obtain

$$\langle\tilde{\boldsymbol{a}}(t)\rangle = e^{-\Gamma t/2}\langle\tilde{\boldsymbol{a}}(0)\rangle. \tag{3.162}$$

Similarly, the occupation number of the oscillator $n \equiv \boldsymbol{a}^\dagger\boldsymbol{a} = \tilde{\boldsymbol{a}}^\dagger\tilde{\boldsymbol{a}}$ decays according to

$$\frac{d}{dt}\langle n\rangle = \frac{d}{dt}\langle\tilde{\boldsymbol{a}}^\dagger\tilde{\boldsymbol{a}}\rangle = tr(\boldsymbol{a}^\dagger\boldsymbol{a}\dot{\boldsymbol{\rho}}_I)$$

$$= \Gamma tr\left(\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{a}\boldsymbol{\rho}_I\boldsymbol{a}^\dagger - \frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho}_I - \frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho}_I\boldsymbol{a}^\dagger\boldsymbol{a}\right)$$
$$= \Gamma tr\boldsymbol{a}^\dagger[\boldsymbol{a}^\dagger,\boldsymbol{a}]\boldsymbol{a}\boldsymbol{\rho}_I = -\Gamma tr\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho}_I = -\Gamma\langle n\rangle, \tag{3.163}$$

which integrates to

$$\langle n(t)\rangle = e^{-\Gamma t}\langle n(0)\rangle. \tag{3.164}$$

Thus Γ is the damping rate of the oscillator. We can interpret the nth excitation state of the oscillator as a state of n noninteracting particles, each with a decay probability Γ per unit time; hence eq. (3.164) is just the exponential law satisfied by the population of decaying particles.

More interesting is what the master equation tells us about decoherence. The details of that analysis will be a homework exercise. But we will analyze here a simpler problem – an oscillator undergoing phase damping.

3.5.4 Phase damping

To model phase damping of the oscillator, we adopt a different coupling of the oscillator to the reservoir:

$$\boldsymbol{H}' = \left(\sum_i \lambda_i \boldsymbol{b}_i^\dagger\boldsymbol{b}_i\right)\boldsymbol{a}^\dagger\boldsymbol{a}. \tag{3.165}$$

Thus, there is just one Lindblad operator, and the master equation in the interaction picture is.

$$\dot{\boldsymbol{\rho}}_I = \Gamma\left(\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho}_I\boldsymbol{a}^\dagger\boldsymbol{a} - \frac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a})^2\boldsymbol{\rho}_I - \frac{1}{2}\boldsymbol{\rho}_I(\boldsymbol{a}^\dagger\boldsymbol{a})^2\right). \tag{3.166}$$

Here Γ can be interpreted as the rate at which reservoir photons are *scattered* when the oscillator is singly occupied. If the occupation number is n then the scattering rate becomes Γn^2. The reason for the factor of n^2 is that the contributions to the scattering amplitude due to each of n oscillator "particles" all add coherently; the amplitude is proportional to n and the rate to n^2.

It is easy to solve for $\dot{\boldsymbol{\rho}}_I$ in the occupation number basis. Expanding

$$\boldsymbol{\rho}_I = \sum_{n,m}\rho_{nm}|n\rangle\langle m|, \tag{3.167}$$

(where $\boldsymbol{a}^\dagger\boldsymbol{a}|n\rangle = n|n\rangle$), the master equation becomes

$$\dot{\rho}_{nm} = \Gamma\left(nm - \frac{1}{2}n^2 - \frac{1}{2}m^2\right)\rho_{nm}$$
$$= -\frac{\Gamma}{2}(n-m)^2\rho_{nm}, \tag{3.168}$$

which integrates to

$$\rho_{nm}(t) = \rho_{nm}(0)\exp\left[-\frac{1}{2}\Gamma t(n-m)^2\right]. \tag{3.169}$$

If we prepare a "cat state" like

$$|\text{cat}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |n\rangle), \quad n \gg 1, \tag{3.170}$$

a superposition of occupation number eigenstates with much different values of n, the off-diagonal terms in the density matrix decay like $\exp(-\frac{1}{2}\Gamma n^2 t)$. In fact, this is just the same sort of behavior we found when we analyzed phase damping for a single qubit. The rate of decoherence is Γn^2 because this is the rate for reservoir photons to scatter off the excited oscillator in the state $|n\rangle$. We also see, as before, that the phase decoherence chooses a preferred basis. Decoherence occurs in the number-eigenstate basis because it is the occupation number that appears in the coupling $\boldsymbol{H}'$ of the oscillator to the reservoir.

Return now to amplitude damping. In our amplitude damping model, it is the annihilation operator $\boldsymbol{a}$ (and its adjoint) that appear in the coupling $\boldsymbol{H}'$ of oscillator to reservoir, so we can anticipate that decoherence will occur in the basis of $\boldsymbol{a}$ eigenstates. The *coherent state*

$$|\alpha\rangle = e^{-|\alpha|^2/2}e^{\alpha a^\dagger}|0\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{3.171}$$

is the normalized eigenstate of $\boldsymbol{a}$ with complex eigenvalue α. Two coherent states with distinct eigenvalues α_1 and α_2 are not orthogonal; rather

$$\begin{aligned} |\langle\alpha_1|\alpha_2\rangle|^2 &= e^{-|\alpha_1|^2}e^{-|\alpha_2|^2}e^{2Re(\alpha_1^*\alpha_2)} \\ &= \exp(-|\alpha_1-\alpha_2|^2), \end{aligned} \tag{3.172}$$

so the overlap is very small when $|\alpha_1-\alpha_2|$ is large.

Imagine that we prepare a cat state

$$|\text{cat}\rangle = \frac{1}{\sqrt{2}}(|\alpha_1\rangle + |\alpha_2\rangle), \tag{3.173}$$

a superposition of coherent states with $|\alpha_1-\alpha_2| \gg 1$. You will show that the off diagonal terms in $\boldsymbol{\rho}$ decay like

$$\exp\left(-\frac{\Gamma t}{2}|\alpha_1-\alpha_2|^2\right) \tag{3.174}$$

(for $\Gamma t << 1$). Thus the decoherence rate

$$\Gamma_{\text{dec}} = \frac{1}{2}|\alpha_1-\alpha_2|^2\Gamma_{\text{damp}}, \tag{3.175}$$

is enormously fast compared to the damping rate. Again, this behavior is easy to interpret. The expectation value of the occupation number in a coherent state is $\langle\alpha|\boldsymbol{a}^\dagger\boldsymbol{a}|\alpha\rangle = |\alpha|^2$. Therefore, if $\alpha_{1,2}$ have comparable moduli but significantly different phases (as for a superposition of minimum uncertainty wave packets centered at x and $-x$), the decoherence rate is of the order of the rate for emission of a *single* photon. This rate is very large compared to the rate for a significant fraction of the oscillator energy to be dissipated.

We can also consider an oscillator coupled to a reservoir with a finite temperature. Again, the decoherence rate is roughly the rate for a single photon to be emitted or absorbed, but the rate is much faster than at zero temperature. Because the photon modes with frequency comparable to the oscillator frequency ω have a thermal occupation number

$$n_\gamma = \frac{T}{\hbar\omega}, \tag{3.176}$$

(for $T \gg \hbar\omega$), the interaction rate is further enhanced by the factor n_γ. We have then

$$\begin{aligned} \frac{\Gamma_{\text{dec}}}{\Gamma_{\text{damp}}} &\sim n_{\text{osc}}n_\gamma \sim \frac{E}{\hbar\omega}\frac{T}{\hbar\omega} \\ &\sim \frac{m\omega^2x^2}{\hbar\omega}\frac{T}{\hbar\omega} \sim x^2\frac{mT}{\hbar^2} \sim \frac{x^2}{\lambda_T^2}, \end{aligned} \tag{3.177}$$

where x is the amplitude of oscillation and λ_T is the thermal de Broglie wavelength. Decoherence is *fast.*

3.6 What is the problem? (Is there a problem?)

Our survey of the foundations of quantum theory is nearly complete. But before we proceed with our main business, let us briefly assess the status of these foundations. Is quantum theory in good shape, or is there a fundamental problem at its roots still in need of resolution?

One potentially serious issue, first visited in §2.1, is the *measurement problem.* We noted the odd dualism inherent in our axioms of quantum theory. There are two ways for the quantum state of a system to change: *unitary* evolution, which is *deterministic*, and *measurement*, which is *probabilistic*. But why should measurement be fundamentally different than any other physical process? The dualism has led some thoughtful people to suspect that our current formulation of quantum theory is still not complete.

In this chapter, we have learned more about measurement. In §3.1.1, we discussed how unitary evolution can establish correlations (entanglement) between a system and the *pointer* of an apparatus. Thus, a pure state of the system can evolve to a mixed state (after we trace over the pointer states), and that mixed state admits an interpretation as an *ensemble* of mutually orthogonal pure states (the eigenstates of the density operator of the system), each occuring with a specified probability. Thus, already in this simple observation, we find the seeds of a deeper understanding of how the "collapse" of a state vector can arise from unitary evolution alone. But on the other hand, we discussed in §2.5 now the ensemble interpretation of a density matrix is ambiguous, and we saw particularly clearly in §2.5.5 that, if we are able to measure the pointer in any basis we please, then we can prepare the system in any one of many "weird" states, superpositions of eigenstates of the system's $\boldsymbol{\rho}$ (the GHJW theorem). Collapse, then (which *destroys* the relative phases of the states in a superposition), cannot be explained by entanglement alone.

In §3.4 and §3.5, we studied another important element of the measurement process – *decoherence*. The key idea is that, for macroscopic systems, we cannot hope to keep track of all microscopic degrees of freedom. We must be content with a *coarse-grained* description, obtained by tracing over the many unobserved variables. In the case of a macroscopic measurement apparatus, we must trace over the degrees of freedom of the environment with which the apparatus inevitably interacts. We then find that the apparatus decoheres exceedingly rapidly in a certain preferred basis, a basis determined by the nature of the coupling of the apparatus to the environment. It seems to be a feature of the Hamiltonian of the world that fundamental interactions are well localized in space, and therefore the basis selected by decoherence is a basis of states that are well localized spatially. The cat is either alive or dead – it is not in the state $1/\sqrt{2}(|\text{Alive}\rangle + |\text{Dead}\rangle)$.

By tracing over the degrees of freedom of the environment, we obtain a more complete picture of the measurement process, of "collapse." Our system becomes entangled with the apparatus, which is in turn entangled with the environment. If we regard the microstate of the environment as forever inaccessible, then we are well entitled to say that a measurement has taken place. The relative phases of the basis states of the system have been lost irrevocably – its state vector has collapsed.

Of course, as a matter of principle, no phase information has really been lost. The evolution of system + apparatus + environment is unitary and deterministic. In principle, we could, perhaps, perform a highly nonlocal measurement of the environment, and restore to the system the phase information that was allegedly destroyed. In this sense, our explanation of collapse is, as John Bell put it, merely FAPP (for all practical purposes). After the "measurement," the coherence of the system basis states could still be restored in principle (we could reverse the measurement by "quantum erasure"), but undoing a measurement is extremely improbable. True, collapse is merely FAPP (though perhaps we might argue, in a cosmological context, that some measurements really *are* irreversible in principle), but isn't FAPP good enough?

Our goal in physics is to account for observed phenomena with a model that is as simple as possible. We should not postulate two fundamental processes (unitary evolution *and* measurement) if only one (unitary evolution) will suffice. Let us then accept, at least provisionally, this hypothesis:

The evolution of a *closed* quantum system is *always unitary*.

Of course, we have seen that not all superoperators are unitary. The point of the hypothesis is that nonunitary evolution in an *open* system, including

the collapse that occurs in the measurement process, always arises from disregarding some of the degrees of freedom of a larger system. This is the view promulgated by Hugh Everett, in 1957. According to this view, the evolution of the quantum state of "the universe" is actually deterministic!

But even if we accept that collapse is explained by decoherence in a system that is truly deterministic, we have not escaped all the puzzles of quantum theory. For the wave function of the universe is in fact a superposition of a state in which the cat is dead and a state in which the cat is alive. Yet each time I look at a cat, it is always either dead or alive. Both outcomes are possible, but only one is realized in *fact*. Why is that?

Your answer to this question may depend on what you think quantum theory is about. There are (at least) two reasonable schools of thought.

Platonic : Physics describes *reality*. In quantum theory, the "wave function of the universe" is a complete description of physical reality.

Positivist : Physics describes our *perceptions*. The wave function encodes our state of knowledge, and the task of quantum theory is to make the best possible predictions about the future, given our current state of knowledge.

I believe in reality. My reason, I think, is a pragmatic one. As a physicist, I seek the most economical model that "explains" what I perceive. To this physicist, at least, the simplest assumption is that my perceptions (and yours, too) are correlated with an underlying reality, external to me. This ontology may seem hopelessly naive to a serious philosopher. But I choose to believe in reality because that assumption seems to be the simplest one that might successfully account for my perceptions. (In a similar vein, I chose to believe that science is more than just a social consensus. I believe that science makes progress, and comes ever closer to a satisfactory understanding of Nature – the laws of physics are discovered, not invented. I believe this because it is the simplest explanation of how scientists are so successful at reaching consensus.)

Those who hold the contrary view (that, even if there is an underlying reality, the state vector only encodes a state of knowledge rather than an underlying reality) tend to believe that the current formulation of quantum theory is not fully satisfactory, that there is a deeper description still awaiting discovery. To me it seems more economical to assume that the wavefunction does describe reality, unless and until you can dissuade me.

If we believe that the wavefunction describes reality and *if* we accept Everett's view that all evolution is unitary, then we must accept that all possible outcomes of a measurement have an equal claim to being "real." How then, are we to understand why, when we do an experiment, only *one* outcome is actually realized – the cat is either alive or dead.

In fact there is no paradox here, but only if we are willing (consistent with the spirit of the Everett interpretation) to include *ourselves* in the quantum system described by the wave function. This wave function describes all the possible correlations among the subsystems, including the correlations between the cat and my mental state. If we prepare the cat state and then look at the cat, the density operator (after we trace over other extraneous degrees of freedom) becomes

$$|\text{Decay}\rangle_{\textbf{atom}}\, |\text{Dead}\rangle_{\text{cat}}\, |\text{Know it's Dead}\rangle_{\text{me}} \quad \left(\text{Prob} = \frac{1}{2}\right)$$

$$|\text{No decay}\rangle_{\textbf{atom}}\, |\text{Alive}\rangle_{\text{cat}}\, |\text{Know it's Alive}\rangle_{\text{me}} \quad \left(\text{Prob} = \frac{1}{2}\right) \tag{3.178}$$

This $\boldsymbol{\rho}$ describes two alternatives, but for either alternative, I am certain about the health of the cat. I *never* see a cat that is half alive and half dead. (I am in an eigenstate of the "certainty operator," in accord with experience.)

By assuming that the wave function describes reality and that all evolution is unitary, we are led to the "many-worlds interpretation" of quantum theory. In this picture, each time there is a "measurement," the wave function of the universe "splits" into two branches, corresponding to the two possible outcomes. After many measurements, there are many branches (many worlds), all with an equal claim to describing reality. This proliferation of worlds seems like an ironic consequence of our program to develop the most economical possible description. But we ourselves follow one particular branch, and for the purpose of predicting what we will see in the next instant, the many other branches are of no consequence. The proliferation of worlds comes at no cost to us. The "many worlds" may seem weird, but should we be surprised if a complete description of reality, something completely foreign to our experience, seems weird to us?

By including ourselves in the reality described by the wave function, we have understood why we perceive a definite outcome to a measurement, but there is still a further question: how does the concept of *probability* enter into this (deterministic) formalism? This question remains troubling, for to answer it we must be prepared to state what is meant by "probability."

The word "probability" is used in two rather different senses. Sometimes *probability* means *frequency*. We say the probability of a coin coming up heads is $1/2$ if we expect, as we toss the coin many times, the number of heads divided by the total number of tosses to converge to $1/2$. (This is a tricky concept though; even if the probability is $1/2$, the coin still *might* come up heads a trillion times in a row.) In rigorous mathematical discussions, probability theory often seems to be a branch of measure theory – it concerns the properties of infinite sequences.

But in everyday life, and also in quantum theory, probabilities typically are *not* frequencies. When we make a measurement, we do not repeat it an infinite number of times on identically prepared systems. In the Everett viewpoint, or in cosmology, there is just one universe, not many identically prepared ones.

So what *is* a probability? In practice, it is a number that quantifies the plausibility of a proposition given a state of knowledge. Perhaps surprisingly, this view can be made the basis of a well-defined mathematical theory, sometimes called the "Bayesian" view of probability. The term "Bayesian" reflects the way probability theory is typically used (both in science and in everyday life) – to test a hypothesis given some observed data. Hypothesis testing is carried out using Bayes's rule for conditional probability

$$P(A_0|B) = P(B|A_0)P(A_0)/P(B). \tag{3.179}$$

For example – suppose that A_0 is the preparation of a particular quantum state, and B is a particular outcome of a measurement of the state. We have made the measurement (obtaining B) and now we want to infer how the state was prepared (compute $P(A_0|B)$. Quantum mechanics allows us to compute $P(B|A_0)$. But it does *not* tell us $P(A_0)$ (or $P(B)$). We have to make a guess of $P(A_0)$, which is possible if we adopt a "principle of indifference" – if we have no knowledge that A_i is more or less likely than A_j we assume $P(A_i) = P(A_j)$. Once an ensemble of preparations is chosen, we can compute

$$P(B) = \sum_i P(B/A_i)P(A_i), \tag{3.180}$$

and so obtain $P(A_0|B)$ by applying Bayes's rule.

But if our attitude will be that probability theory quantifies plausibility given a state of knowledge, we are obligated to ask "whose state of knowledge?" To recover an objective theory, we must interpret probability in quantum theory not as a prediction based on our *actual* state of knowledge, but rather as a prediction based on the most complete *possible* knowledge about the quantum state. If we prepare $|\uparrow_x\rangle$ and measure $\boldsymbol{\sigma}_3$, then we say that the result is $|\uparrow_z\rangle$ with probability 1/2, not because that is the best prediction we can make based on what we know, but because it is the best prediction *anyone* can make, no matter how much they know. It is in this sense that the outcome is truly *random*; it cannot be predicted with certainty even when our knowledge is complete (in contrast to the pseudo randomness that arises in classical physics because our knowledge is incomplete).

So how, now, are we to extract probabilities from Everett's deterministic universe? Probabilities arise because we (a part of the system) cannot predict our future with certainty. I know the formalism, I know the Hamiltonian and wave function of the universe, I know my branch of the wave function. Now I am about to look at the cat. A second from now, I will be either be certain that the cat is dead or I will be certain that it is alive. Yet even with all I know, I cannot predict the future. Even with complete knowledge about the present, I cannot say what my state of knowledge will be after I look at the cat. The best I can do is assign probabilities to the outcomes. So, while the wave function of the universe *is* deterministic I, as a part of the system, can do no better than making probabilistic predictions.

Of course, as already noted, decoherence is a crucial part of this story. We may consistently assign probabilities to the alternatives Dead and Alive only if there is no (or at least negligible) possibility of interference among the alternatives. Probabilities make sense only when we can identify an exhaustive set of mutually exclusive alternatives. Since the issue is really whether interference might arise at a later time, we cannot decide whether probability theory applies by considering a quantum state at a fixed time; we must examine a set of mutually exclusive (coarse-grained) histories, or sequences of events. There is a sophisticated technology ("decoherence functionals") for adjudicating whether the various histories decohere to a sufficient extent for probabilities to be sensibly assigned.

So the Everett viewpoint *can* be reconciled with the quantum indeterminism that we observe, but there is still a troubling gap in the picture, at least as far as I can tell. I am about to look at the cat, and I know that the density matrix a second from now will be

$$|\text{Dead}\rangle_{\text{cat}}\ |\text{Know it's Dead}\rangle_{\text{me}}\ , \quad \text{Prob} = p_{\text{dead}},$$

$$|\text{Alive}\rangle_{\text{cat}}\ |\text{Know it's Alive}\rangle_{\text{me}}\ , \quad \text{Prob} = p_{\text{alive}}. \tag{3.181}$$

But how do I infer that p_{dead} and p_{alive} actually are probabilities that I (in my Bayesian posture) may assign to my future perceptions? I *still* need a rule to translate this density operator into probabilities assigned to the alternatives. It seems contrary to the Everett philosophy to *assume* such a rule; we could prefer to say that the only rule needed to define the theory is the Schrödinger equation (and perhaps a prescription to specify the initial wave function). Postulating a probability formula comes perilously close to allowing that there is a nondeterministic measurement process after all. So here is the issue regarding the foundations of theory for which I do not know a fully satisfying resolution.

Since we have not been able to remove all discomfiture concerning the origin of probability in quantum theory, it may be helpful to comment on an interesting suggestion due to Hartle. To implement his suggestion, we must return (perhaps with regret) to the frequency interpretation of probability. Hartle's insight is that we need not assume the probability interpretation as part of the measurement postulate. It is really sufficient to make a weaker assumption:

> If we prepare a quantum state $|a\rangle$, such that $\mathbf{A}|a\rangle = a|a\rangle$, and then immediately measure $\mathbf{A}$, the outcome of the measurement is a.

This seems like an assumption that a Bayesian residing in Everett's universe would accept. I am about to measure an observable, and the wavefunction will branch, but if the observable has the *same* value in every branch, then I *can* predict the outcome.

To implement a frequency interpretation of probability, we should, strictly speaking, consider an infinite number of trials. Suppose we want to make a statement about the probability of obtaining the result $|\uparrow_z\rangle$ when we measure $\boldsymbol{\sigma}_3$ in the state

$$|\psi\rangle = a|\uparrow_z\rangle + b|\downarrow_z\rangle. \tag{3.182}$$

Then we should imagine that an infinite number of copies are prepared, so the state is

$$|\psi^{(\infty)}\rangle \equiv (|\psi\rangle)^{\infty} = |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \otimes \cdots \tag{3.183}$$

and we imagine measuring $\boldsymbol{\sigma}_3$ for each of the copies. Formally, the case of an infinite number of trials can be formulated as the $N \to \infty$ limit of N trials.

Hartle's idea is to consider an "average spin" operator

$$\bar{\boldsymbol{\sigma}}_3 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{\sigma}_3^{(i)}, \tag{3.184}$$

and to argue that $(|\psi\rangle)^N$ becomes an *eigenstate* of $\bar{\boldsymbol{\sigma}}_3$ with eigenvalue $|a|^2 - |b|^2$, as $N \to \infty$. Then we can invoke the weakened measurement postulate to infer that a measurement of $\bar{\boldsymbol{\sigma}}_3$ will yield the result $|a|^2 - |b|^2$ with certainty, and that the fraction of all the spins that point up is therefore $|a|^2$. In this sense, $|a|^2$ is the probability that the measurement of $\boldsymbol{\sigma}_3$ yields the outcome $|\uparrow_z\rangle$.

Consider, for example, the special case

$$|\psi_x^{(N)}\rangle \equiv (|\uparrow_x\rangle)^N = \left[\frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle)\right]^N. \tag{3.185}$$

We can compute

$$\begin{aligned}
&\langle\psi_x^{(N)}|\bar{\boldsymbol{\sigma}}_3|\psi_x^{(N)}\rangle = 0\ , \\
&\langle\psi_x^{(N)}|\bar{\boldsymbol{\sigma}}_3^2|\psi_x^{(N)}\rangle \\
&= \frac{1}{N^2}\langle\psi_x^{(N)}|\sum_{ij}\boldsymbol{\sigma}_3^{(i)}\boldsymbol{\sigma}_3^{(j)}|\psi_x^{(N)}\rangle \\
&= \frac{1}{N^2}\sum_{ij}\delta^{ij} = \frac{N}{N^2} = \frac{1}{N}.
\end{aligned} \tag{3.186}$$

Taking the formal $N \to \infty$ limit, we conclude that $\bar{\boldsymbol{\sigma}}_3$ has vanishing dispersion about its mean value $\langle\bar{\boldsymbol{\sigma}}_3\rangle = 0$, and so at least in this sense $|\psi_x^{(\infty)}\rangle$ is an "eigenstate" of $\bar{\boldsymbol{\sigma}}_3$ with eigenvalue zero.

Coleman and Lesniewski have noted that one can take Hartle's argument a step further, and argue that the measurement outcome $|\uparrow_z\rangle$ not only occurs with the right frequency, but also that the $|\uparrow_z\rangle$ outcomes are *randomly distributed*. To make sense of this statement, we must formulate a definition of randomness. We say that an infinite string of bits is random if the string is *incompressible*; there is no simpler way to generate the first N bits than simply writing them out. We formalize this idea by considering the length

of the shortest computer program (on a certain universal computer) that generates the first N bits of the sequence. Then, for a random string

$$\text{Length of shortest program } > N - \text{ const.} \tag{3.187}$$

where the constant may depend on the particular computer used or on the particular sequence, but not on N.

Coleman and Lesniewski consider an orthogonal projection operator $\boldsymbol{E}_{\text{random}}$ that, acting on a state $|\psi\rangle$ that is an eigenstate of each $\boldsymbol{\sigma}_3^{(i)}$, satisfies

$$\boldsymbol{E}_{\text{random}}|\psi\rangle = |\psi\rangle, \tag{3.188}$$

if the sequence of eigenvalues of $\boldsymbol{\sigma}_3^{(i)}$ is random, and

$$\boldsymbol{E}_{\text{random}}|\psi\rangle = 0, \tag{3.189}$$

if the sequence is not random. This property alone is not sufficient to determine how $\boldsymbol{E}_{\text{random}}$ acts on all of $(\mathcal{H}_2)^\infty$, but with an additional technical assumption, they find that $\boldsymbol{E}_{\text{random}}$ exists, is unique, and has the property

$$\boldsymbol{E}_{\text{random}}|\psi_x^{(\infty)}\rangle = |\psi_x^{(\infty)}\rangle. \tag{3.190}$$

Thus, we "might as well say" that $|\psi_x^{(\infty)}\rangle$ is random, with respect to $\boldsymbol{\sigma}_3$ measurements – a procedure for distinguishing the random states from nonrandom ones that works properly for strings of $\boldsymbol{\sigma}_3$ eigenstates, will inevitably identify $|\psi_x^{(\infty)}\rangle$ as random, too.

These arguments are interesting, but they do not leave me completely satisfied. The most disturbing thing is the need to consider infinite sequences (a feature of any frequency interpretation probability). For any finite N, we are unable to apply Hartle's weakened measurement postulate, and even in the limit $N \to \infty$, applying the postulate involves subtleties. It would be preferable to have *a stronger* weakened measurement postulate that could be applied at finite N, but I am not sure how to formulate that postulate or how to justify it.

In summary then: Physics should describe the objective physical world, and the best representation of physical reality that we know about is the quantum-mechanical wave function. Physics should aspire to explain all observed phenomena as economically as possible – it is therefore unappealing to postulate that the measurement process is governed by different dynamical principles than other processes. Fortunately, everything we know about physics is compatible with the hypothesis that all physical processes (including measurements) can be accurately modeled by the unitary evolution of a wave function (or density matrix). When a microscopic quantum system interacts with a macroscopic apparatus, decoherence drives the "collapse" of the wave function "for all practical purposes."

If we eschew measurement as a mystical primitive process, and we accept the wave function as a description of physical reality, then we are led to the Everett or "many-worlds" interpretation of quantum theory. In this view, all possible outcomes of any "measurement" are regarded as "real" — but I perceive only a specific outcome because the state of my brain (a part of the quantum system) is strongly correlated with the outcome.

Although the evolution of the wave function in the Everett interpretation is deterministic, I am unable to predict with certainty the outcome of an experiment to be performed in the future – I don't know what branch of the wavefunction I will end up on, so I am unable to predict my future state of mind. Thus, while the "global" picture of the universe is in a sense deterministic, from my own local perspective from within the system, I perceive quantum mechanical randomness.

My own view is that the Everett interpretation of quantum theory provides a satisfying explanation of measurement and of the origin of randomness, but does not yet fully explain the quantum mechanical rules for computing probabilities. A full explanation should go beyond the frequency interpretation of probability — ideally it would place the Bayesian view of probability on a secure objective foundation.

3.7 Summary

POVM. If we restrict our attention to a subspace of a larger Hilbert space, then an orthogonal (Von Neumann) measurement performed on the larger space cannot in general be described as an orthogonal measurement on the subspace. Rather, it is a *generalized measurement* or *POVM* – the outcome a occurs with a probability

$$\mathrm{Prob}(a) = tr\left(\boldsymbol{F}_a\boldsymbol{\rho}\right) \ , \tag{3.191}$$

where $\boldsymbol{\rho}$ is the density matrix of the subsystem, each $\boldsymbol{F}_a$ is a positive hermitian operator, and the $\boldsymbol{F}_a$'s satisfy

$$\sum_a \boldsymbol{F}_a = \mathbf{1} \ . \tag{3.192}$$

A POVM in $\mathcal{H}_A$ can be realized as a unitary transformation on the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$, followed by an orthogonal measurement in $\mathcal{H}_B$.

Superoperator. Unitary evolution on $\mathcal{H}_A \otimes \mathcal{H}_B$ will not in general appear to be unitary if we restrict our attention to $\mathcal{H}_A$ alone. Rather, evolution in $\mathcal{H}_A$ will be described by a *superoperator*, (which can be inverted by another superoperator only if unitary). A general superoperator \$ has an operator-sum (Kraus) representation:

$$\$: \boldsymbol{\rho} \to \$(\boldsymbol{\rho}) = \sum_\mu \boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger \ , \tag{3.193}$$

where

$$\sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \mathbf{1} \ . \tag{3.194}$$

In fact, any reasonable (linear and completely positive) mapping of density matrices to density matrices has unitary and operator-sum representations.

Decoherence. Decoherence – the decay of quantum information due to the interaction of a system with its environment – can be described by a superoperator. If the environment frequently "scatters" off the system, and the state of the environment is not monitored, then off-diagonal terms in the density matrix of the system decay rapidly in a preferred basis (typically a spatially localized basis selected by the nature of the coupling of the system to the environment). The time scale for decoherence is set by the scattering rate, which may be much larger than the damping rate for the system.

Master Equation. When the relevant dynamical time scale of an open quantum system is long compared to the time for the environment to "forget" quantum information, the evolution of the system is effectively local in time (the Markovian approximation). Much as general unitary evolution is generated by a Hamiltonian, a general Markovian superoperator is generated by a *Lindbladian* $\mathcal{L}$ as described by the *master equation*:

$$\dot{\boldsymbol{\rho}} \equiv \mathcal{L}[\boldsymbol{\rho}] = -i[\boldsymbol{H}, \boldsymbol{\rho}] + \sum_\mu \left(\boldsymbol{L}_\mu \boldsymbol{\rho} \boldsymbol{L}_\mu^\dagger - \frac{1}{2}\boldsymbol{L}_\mu^\dagger \boldsymbol{L}_\mu \boldsymbol{\rho} - \frac{1}{2}\boldsymbol{\rho}\boldsymbol{L}_\mu^\dagger \boldsymbol{L}_\mu \right) . \tag{3.195}$$

Here each *Lindblad operator* (or *quantum jump operator*) represents a "quantum jump" that could in principle be detected if we monitored the environment faithfully. By solving the master equation, we can compute the decoherence rate of an open system.

3.8 Exercises

3.1 Realization of a POVM

Consider the POVM defined by the four positive operators

$$\begin{aligned} P_1 &= \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| \,, \quad P_2 = \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z| \\ P_3 &= \frac{1}{2}|\uparrow_x\rangle\langle\uparrow_x| \,, \quad P_4 = \frac{1}{2}|\downarrow_x\rangle\langle\downarrow_x| \,. \end{aligned} \tag{3.196}$$

Show how this POVM can be realized as an orthogonal measurement in a two-qubit Hilbert space, if one ancilla spin is introduced.

3.2 Invertibility of superoperators

The purpose of this exercise is to show that a superoperator is invertible only if it is unitary. Recall that any superoperator has an operator-sum representation; it acts on a pure state as

$$\mathcal{M}(|\psi\rangle\langle\psi|) = \sum_\mu \mathbf{M}_\mu|\psi\rangle\langle\psi|\mathbf{M}_\mu^\dagger, \tag{3.197}$$

where $\sum_\mu \mathbf{M}_\mu^\dagger \mathbf{M}_\mu = \mathbf{1}$. Another superoperator $\mathcal{N}$ is said to be the inverse of $\mathcal{M}$ if $\mathcal{N}\circ\mathcal{M} = I$, or

$$\sum_{\mu,a} \mathbf{N}_a\mathbf{M}_\mu|\psi\rangle\langle\psi|\mathbf{M}_\mu^\dagger\mathbf{N}_a^\dagger = |\psi\rangle\langle\psi|, \tag{3.198}$$

for any $|\psi\rangle$. It follows that

$$\sum_{\mu,a} |\langle\psi|\mathbf{N}_a\mathbf{M}_\mu|\psi\rangle|^2 = 1. \tag{3.199}$$

a) Show, using the normalization conditions satisfied by the $\mathbf{N}_a$'s and $\mathbf{M}_\mu$'s, that $\mathcal{N}\circ\mathcal{M} = I$ implies that

$$\mathbf{N}_a\mathbf{M}_\mu = \lambda_{a\mu}\mathbf{1}, \tag{3.200}$$

for *each* a and μ; i.e., that each $\mathbf{N}_a\mathbf{M}_\mu$ is a multiple of the identity.

b) Use the result of (a) to show that $\mathbf{M}_\nu^\dagger\mathbf{M}_\mu$ is proportional to the identity for each μ and ν.

c) Show that it follows from (b) that $\mathcal{M}$ is unitary.

3.3 How many superoperators?

How many real parameters are needed to parametrize the general superoperator

$$\$: \boldsymbol{\rho} \to \boldsymbol{\rho}' \,, \tag{3.201}$$

if $\boldsymbol{\rho}$ is a density operator in a Hilbert space of dimension N? [Hint: How many real parameters parametrize an $N\times N$ Hermitian matrix? How many for a linear mapping of Hermitian matices to Hermitian matrices? How many for a *trace-preserving* mapping of Hermitian matrices to Hermitian matrices?]

3.4 How fast is decoherence?

A very good pendulum with mass $m = 1$ g and circular frequency $\omega = 1\ \mathrm{s}^{-1}$ has quality factor $Q = 10^9$. The pendulum is prepared in the "cat state"

$$|\mathrm{cat}\rangle = \frac{1}{\sqrt{2}}(|x\rangle + |-x\rangle), \tag{3.202}$$

a superposition of minimum uncertainty wave packets, each initially at rest, centered at positions $\pm x$, where $x = 1$ cm. Estimate, in order of magnitude, how long it takes for the cat state to decohere, if the environment is at

a) zero temperature.

b) room temperature.

3.5 Phase damping

In class, we obtained an operator-sum representation of the phase-damping channel for a single qubit, with Kraus operators

$$\mathbf{M}_0 = \sqrt{1-p}\,\mathbf{1}, \quad \mathbf{M}_1 = \sqrt{p}\,\frac{1}{2}(\mathbf{1}+\boldsymbol{\sigma}_3),$$

$$\mathbf{M}_2 = \sqrt{p}\,\frac{1}{2}(\mathbf{1}-\boldsymbol{\sigma}_3). \tag{3.203}$$

a) Find an alternative representation using only two Kraus operators $\mathbf{N}_0, \mathbf{N}_1$.

b) Find a unitary 3×3 matrix $U_{\mu a}$ such that your Kraus operators found in (a) (augmented by $\mathbf{N}_2 = 0$) are related to $\mathbf{M}_{0,1,2}$ by

$$\mathbf{M}_\mu = U_{\mu a}\mathbf{N}_a. \tag{3.204}$$

c) Consider a single-qubit channel with a unitary representation

$$\begin{aligned}|0\rangle_A|0\rangle_E &\to \sqrt{1-p}\ |0\rangle_A|0\rangle_E + \sqrt{p}\ |0\rangle_A|\gamma_0\rangle_E \\ |1\rangle_A|0\rangle_E &\to \sqrt{1-p}\ |1\rangle_A|0\rangle_E + \sqrt{p}\ |1\rangle_A|\gamma_1\rangle_E,\end{aligned} \tag{3.205}$$

where $|\gamma_0\rangle_E$ and $|\gamma_1\rangle_E$ are normalized states, both orthogonal to $|0\rangle_E$, that satisfy

$$_E\langle\gamma_0|\gamma_1\rangle_E = 1 - \varepsilon, \quad 0 < \varepsilon < 1. \tag{3.206}$$

Show that this is again the phase-damping channel, and find its operator-sum representation with two Kraus operators.

d) Suppose that the channel in (c) describes what happens to the qubit when a single photon scatters from it. Find the decoherence rate Γ_{decoh} in terms of the scattering rate Γ_{scatt}.

3.6 Decoherence on the Bloch sphere

Parametrize the density matrix of a single qubit as

$$\boldsymbol{\rho} = \frac{1}{2}\left(\mathbf{1} + \vec{P}\cdot\vec{\boldsymbol{\sigma}}\right). \tag{3.207}$$

a) Describe what happens to $\vec{P}$ under the action of the phase-damping channel.

b) Describe what happens to $\vec{P}$ under the action of the amplitude-damping channel defined by the Kraus operators.

$$\mathbf{M}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p}\end{pmatrix}, \quad \mathbf{M}_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0\end{pmatrix}. \tag{3.208}$$

c) The same for the "two-Pauli channel:"

$$\mathbf{M}_0 = \sqrt{1-p}\ \mathbf{1}, \quad \mathbf{M}_1 = \sqrt{\frac{p}{2}}\ \boldsymbol{\sigma}_1, \quad \mathbf{M}_2 = \sqrt{\frac{p}{2}}\ \boldsymbol{\sigma}_3. \tag{3.209}$$

3.7 Decoherence of the damped oscillator

We saw in class that, for an oscillator that can emit quanta into a zero-temperature reservoir, the interaction picture density matrix $\boldsymbol{\rho}_I(t)$ of the oscillator obeys the master equation

$$\dot{\boldsymbol{\rho}}_I = \Gamma\left(\boldsymbol{a}\boldsymbol{\rho}_I\boldsymbol{a}^\dagger - \frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho}_I - \frac{1}{2}\boldsymbol{\rho}_I\boldsymbol{a}^\dagger\boldsymbol{a}\right), \tag{3.210}$$

where $\boldsymbol{a}$ is the annihilation operator of the oscillator.

a) Consider the quantity

$$X(\lambda, t) = tr\left[\boldsymbol{\rho}_I(t)e^{\lambda\boldsymbol{a}^\dagger}e^{-\lambda^*\boldsymbol{a}}\right], \tag{3.211}$$

(where λ is a complex number). Use the master equation to derive and solve a differential equation for $X(\lambda, t)$. You should find

$$X(\lambda, t) = X(\lambda', 0), \tag{3.212}$$

where λ' is a function of λ, Γ, and t. What is this function $\lambda'(\lambda, \Gamma, t)$?

b) Now suppose that a "cat state" of the oscillator is prepared at $t = 0$:

$$|\text{cat}\rangle = \frac{1}{\sqrt{2}}\left(|\alpha_1\rangle + |\alpha_2\rangle\right), \tag{3.213}$$

where $|\alpha\rangle$ denotes the coherent state

$$|\alpha\rangle = e^{-|\alpha|^2/2}e^{\alpha\boldsymbol{a}^\dagger}|0\rangle. \tag{3.214}$$

Use the result of (a) to infer the density matrix at a later time t. Assuming $\Gamma t \ll 1$, at what rate do the off-diagonal terms in $\boldsymbol{\rho}$ decay (in this coherent state basis)?