ANOMALIES AND FERMION MASSES IN D DIMENSIONS

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We show that anomalies in D spacetime dimensions are generated by only *massless* fermions, and note that the manner in which the anomalies of *massive* fermions cancel is qualitatively different in the two cases $D \equiv 2$ and $D \equiv 0 \pmod 4$.

Recently, attention has been focused on the anomalies of D-dimensional theories of the Kaluza–Klein type, in which $D - 4$ spacetime dimensions are compactified. In particular, gauge-invariance requires the underlying D-dimensional theory to be anomaly-free, and it has been suggested [1] that this requirement might constrain the representation content under the effective low-energy gauge group of those fermions which survive in the effective four-dimensional theory. Underlying this suggestion is the conjecture that the fermions which acquire large masses and are removed from the four-dimensional effective theory must have vanishing D-dimensional anomalies.

We will not prove this conjecture here, but will instead address a simpler question. We will show that fermions which acquire masses in a D-dimensional field theory necessarily give a vanishing contribution to the anomalies of the unbroken gauge-symmetry currents. The nature of this anomaly cancellation will be seen to be qualitatively different for the case $D \equiv 2 \pmod 4$ than for the more familiar case $D \equiv 0 \pmod 4$. We do not consider here the effect of dimensional compactification on the fermion content of the theory.

Let us begin by recalling the familiar connection between anomaly cancellation and fermion masses in $D = 4$ dimensions. If the fermions of an underlying gauge theory are chosen so that anomalies cancel, then the light fermions in the effective low-energy theory must also have vanishing anomalies under the surviving unbroken gauge group. Equivalently, the fermions which acquire large masses have vanishing anomalies under the surviving gauge group. This result can be stated in a somewhat more general way: Suppose that G is a group of (gauged or global) symmetries of a field theory which are exact up to anomalies, and that some of the fermions in the theory have G-invariant masses. Then the massive fermions must have trivial G-anomalies.

This result is easily demonstrated. It is convenient to choose all the fermions to be left-handed Weyl spinors (eigenstates of γ_5 with eigenvalue +1). Those fermions with G-invariant masses must transform as a real (or pseudoreal) representation R of G. But the G anomaly due to these massive fermions is proportional to

$$A^{abc} = \tfrac{1}{2}\,\mathrm{tr}\,\{\lambda^a, \lambda^b\}\,\lambda^c\,, \qquad (1)$$

where the λ^a are the G generators in the representa-

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tion R. If the fermion representation R is real (or pseudoreal), then there exists a unitary matrix U such that

$$(\lambda^a)^* = -U\lambda^a U^{-1} , \tag{2}$$

and, since A^{abc} is real, we conclude that

$$A^{abc} = -A^{abc} = 0 . \tag{3}$$

In short, the massive fermions are in a real (or pseudoreal) representation of G, if their mass terms are G-invariant, and fermions in a real (or pseudoreal) representation of G give a vanishing contribution to the G anomaly.

Now let us consider how this argument generalizes to D dimensions. Anomalies occur only if D is even, so we will restrict our attention to that case. For any even D, there is a γ_{D+1}, analogous to γ_5 in four dimensions, which commutes with the Lorentz generators (see below), so we may define Weyl spinors which are eigenstates of γ_{D+1} with eigenvalue ± 1. In D dimension, anomalies are generated by l-gon graphs, where $l = D/2 + 1$, and the G anomaly of a Weyl spinor in the representation R of G is proportional to the group-theoretic factor

$$A^{a_1 \ldots a_l} = \frac{1}{l!} \sum_{\text{perms}} \operatorname{tr}(\lambda^{a_1} \ldots \lambda^{a_l}) , \tag{4}$$

where the λ^a are the G generators in the representation R. If l is odd we may argue as before that Weyl fermions in a real (or pseudoreal) representation of G have a vanishing G anomaly. But if l is even, this argument fails. "Left-handed" ($\gamma_{D+1} = +1$) Weyl fermions in a complex representation R of G give the same contribution to the G anomaly as left-handed fermions in the conjugate fermion $\bar{\mathrm{R}}$, if $D = 4n + 2$, where n is an integer.

It is nonetheless true that fermions with G-invariant masses cannot contribute to the G anomaly, for any even D. We will next show that this conclusion follows from a simple analyticity argument. Then we will discuss the structure of Lorentz-invariant fermion mass terms in D dimensions, to see how this conclusion can be consistent with the observation that, if $D = 4n + 2$, left-handed Weyl fermions in the representations R and $\bar{\mathrm{R}}$ of G have equal G anomalies.

The argument which shows that fermions with G-invariant masses cannot contribute to the G anomaly is a trivial generalization of the argument presented for the $D = 4$ case in ref. [2]. Let J^μ be any one of the G currents, which may be written in terms of the $2^{D/2}$-component Dirac spinors ψ^a as

$$J^\mu = \bar{\psi}^a [A_+^{ab} \tfrac{1}{2}(1 + \gamma_{D+1}) + A_-^{ab} \tfrac{1}{2}(1 - \gamma_{D+1})] \gamma^\mu \psi , \tag{5}$$

and consider the l-point function ($l = D/2 + 1$)

$$\Gamma_{\mu_1 \ldots \mu_l}(p_1, \ldots, p_l) \delta^D(p_1 + \ldots + p_l) = \int \mathrm{d}^D x_1 \ldots \mathrm{d}^D x_l \langle 0 | T[J_{\mu_1}(x_1) \ldots J_{\mu_l}(x_l)] | 0 \rangle \times \exp(\mathrm{i}p_1 x_1) \ldots \exp(\mathrm{i}p_l x_l) . \tag{6}$$

The D-dimensional anomaly equation, satisfied if J^μ is exactly conserved up to anomalies, is:

$$p_l^{\mu_l} \Gamma_{\mu_1 \ldots \mu_l} = c_l \operatorname{tr}(A_+^l - A_-^l) \epsilon_{\mu_1 \ldots \mu_{l-1}, \alpha_1, \ldots \alpha_{l-1}} p_1^{\alpha_1} \ldots p_{l-1}^{\alpha_{l-1}} . \tag{7}$$

Now it is easily seen that the term in Γ which gives rise to the anomaly must be singular at zero momentum. Because any term in Γ which is analytic at the point $p_1 = p_2 = \ldots = p_l = 0$ can be expanded in a Taylor series about that point. Since the anomaly [the right-hand side of eq. (7)] is a polynomial in the momenta of degree $l - 1$, an analytic term in Γ contributing to the anomaly is a polynomial of degree $l - 2$, which must have the form

$$\Gamma_{\mu_1 \ldots \mu_l} = C \epsilon_{\mu_1 \ldots \mu_l, \alpha_1 \ldots \alpha_{l-2}} p_1^{\alpha_1} \ldots p_{l-2}^{\alpha_{l-2}} + \text{crossed terms} . \tag{8}$$

But, since the l currents appearing in eq. (6) are identical, Γ must be completely crossing-symmetric, and there is no expression of the form (8) with complete crossing symmetry. [For example, if we attempt to symmetrize the term appearing in (8) under $(p_{l-1}, \mu_{l-1}) \leftrightarrow (p_l, \mu_l)$, we obtain zero.] It follows that no term in Γ due to massive fermions, which is necessarily analytic at zero momentum, has no effect on the anomaly. (Of ocurse, it is necessary for the fermion masses to be G-invariant, for we have required the G currents to be conserved up to anomalies.)

To understand how the G anomalies of fermions with G-invariant masses cancel for $D = 4n + 2$, we must consider the structure of Lorentz-invariant fermion mass terms in D dimensions. (Properties of spinors in D dimensions are discussed in ref. [3].) We will treat here only the case of even D. In D-spacetime dimensions, the Dirac matrices γ^μ are $2^{D/2} \times 2^{D/2}$ matrices satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} , \tag{9}$$

where $\eta^{\mu\nu}$ has the signature $(+, -, ..., -)$. A $2^{D/2}$-component Dirac spinor ψ transforms under an infinitesimal Lorentz transformation parametrized by $\epsilon_{\mu\nu}$ as

$$\delta\psi = \tfrac{1}{2}\epsilon_{\mu\nu}\Sigma^{\mu\nu}\psi ,$$

where

$$\Sigma^{\mu\nu} = \tfrac{1}{4}[\gamma^\mu, \gamma^\nu] . \tag{10}$$

Thus, the bilinear $\tilde{\psi}\psi$ is Lorentz-invariant if

$$\tilde{\psi} = \psi^{\mathrm{T}} A \quad \text{and} \quad A^{-1}(\Sigma^{\mu\nu})^{\mathrm{T}} A = -\Sigma^{\hat{\mu}\nu} . \tag{11}$$

Furthermore, a charge conjugate of ψ can be defined,

$$\psi^{\mathrm{c}} = C^{-1}\psi^* , \tag{12}$$

which transforms as ψ does provided that

$$C^{-1}(\Sigma^{\mu\nu})^* C = \Sigma^{\mu\nu} . \tag{13}$$

Therefore, $\tilde{\psi}\psi^{\mathrm{c}}$, $(\tilde{\psi}^{\mathrm{c}})\psi$ and $(\tilde{\psi}^{\mathrm{c}})\psi^{\mathrm{c}}$ are also Lorentz-invariant bilinears.

For any even D we may define a γ_{D+1}

$$\gamma_{D+1} = (\mathrm{i})^{D/2-1}\gamma^0\gamma^1 \dots \gamma^{D-1} , \tag{14}$$

which commutes with all the $\Sigma^{\mu\nu}$. Therefore, the Dirac spinor decomposes into two irreducible representations of the Lorentz group which are eigenstates of γ_{D+1}. The phase in eq. (14) has been chosen so that $\gamma_{D+1}^2 = 1$, and the eigenvalues are therefore ± 1. The corresponding eigenstates, the $2^{D/2-1}$ dimensional Weyl spinors, will be denoted ψ_+ and ψ_-.

Let us now express the Lorentz-invariant bilinears in terms of the Weyl spinors ψ_+ and ψ_-, Although the A and C satisfying (11) and (13) are not unique, their commutation properties with γ_{D+1} *are* determined by (11) and (13). We may write γ_{D+1} as

$$\gamma_{D+1} = (2)^{D/2}(\mathrm{i})^{D/2-1}\Sigma^{01}\Sigma^{23} \dots \Sigma^{D-2,D-1} , \tag{15}$$

from which we deduce

$$A^{-1}(\gamma_{D+1})^{\mathrm{T}} A = (-1)^{D/2}\gamma_{D+1} ,$$
$$C^{-1}(\gamma_{D+1})^* C = (-1)^{D/2-1}\gamma_{D+1} . \tag{16}$$

If we define projection operators $P(\pm) = \frac{1}{2}(1 \pm \gamma_{D+1})$, such that $\psi_\pm = P(\pm)\psi$, it follows from (16) that

$$\tilde{\psi}_\pm = \tilde{\psi}P[\pm(-1)^{D/2}]$$
$$(\psi_\pm)^{\mathrm{c}} = P[\pm(-1)^{D/2-1}]\,\psi^{\mathrm{c}} . \tag{17}$$

Therefore, if $D = 4n$, charge conjugation flips the chirality (eigenvalue of γ_{D+1}) of a Weyl spinor, and there are two distinct types of fermion mass terms – "Majorana" mass terms $\tilde{\psi}_+\psi_+$ coupling spinors of the same chirality, and "Dirac" mass terms $(\tilde{\psi}^{\mathrm{c}}_-)\psi_+$ coupling spinors of opposite chirality. But for $D = 4n + 2$, charge conjugation does not flip the chirality of a Weyl spinor, and the only Lorentz-invariant bilinears $\tilde{\psi}_-\psi_+$ and $(\tilde{\psi}^{\mathrm{c}}_-)\psi_+$ couple spinors of opposite chirality.

If $D = 4n$, then, a Weyl spinor ψ_+ transforming as the representation R of the symmetry group G is equivalent to the spinor ψ_- transforming as the conjugate representation $\bar{\mathrm{R}}$; one may be transformed into the other by charge conjugation. We have the freedom to choose all the spinors to have chirality + 1. If $D = 4n + 2$, there is no such freedom. Instead, ψ_+ transforming as the representation R is equivalent to ψ_+ transforming as $\bar{\mathrm{R}}$; charge conjugation does not flip chirality. The G anomaly of ψ_+ can be cancelled only by a spinor of opposite chirality; in particular, the G anomaly of ψ_+ transforming as R may be cancelled by ψ_- transforming as either R or $\bar{\mathrm{R}}$. Since a Lorentz-invariant and G-invariant mass term necessarily couples ψ_+ transforming as R to ψ_- transforming as either R or R, fermions with G-invariant masses always give cancelling contributions to the G anomaly.

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